MA 262

ASSIGNMENT 1 KEY

I.a) (3) $\int x^{-2} \ln x \, dx$

$u = \ln x \qquad dv = x^{-2}dx$

$du = \frac{dx}{x} \qquad v = -x^{-1}$

$$\int x^{-2}\ln x\,dx = -\frac{1}{x}\ln x + \int x^{-1}\cdot\frac{dx}{x} = -\frac{1}{x}\ln x + \int x^{-2}dx = -\frac{1}{x}\ln x - \frac{1}{x} + C$$

$$= -\frac{1}{x}(\ln x + 1) + C$$

Check: $\frac{d}{dx}\left\{-\frac{1}{x}(\ln x + 1) + C\right\} = +\frac{1}{x^2}(\ln x + 1) - \frac{1}{x^2} = \frac{\ln x}{x^2}$

b.) (4) $\int \frac{x^3 e^{x^2}}{(x^2+1)^2}dx$

Method 1: $u = x^2 \quad du = 2x\,dx$

$$\therefore \int \frac{x^3 e^{x^2} dx}{(x^2+1)^2} = \frac{1}{2}\int \frac{u e^u du}{(u+1)^2} = \frac{1}{2}\int \frac{(w-1)}{w^2} e^{w-1} dw$$

$$= \frac{1}{2e}\int [w^{-1} - w^{-2}] e^w dw = \frac{1}{2e}\left\{\int \underbrace{w^{-1}}_{u}\underbrace{e^w dw}_{dv} - \int w^{-2}e^w dw\right\}$$

$$= \frac{1}{2e}\left\{w^{-1}e^w - \int(-w^{-2}dw)e^w - \int w^{-2}e^w dw\right\}$$

$$= \frac{1}{2e}\left\{w^{-1}e^w + \int w^{-2}e^w dw - \int w^{-2}e^w dw\right\} = \frac{1}{2e}\frac{e^w}{w} = \frac{1}{2e}\frac{e^{u+1}}{(u+1)} + C$$

$$= \frac{e^{x^2}}{2(x^2+1)} + C = \frac{1}{2}(x^2+1)^{-1}e^{x^2} + C$$

Check: $\frac{d}{dx}\left\{\frac{1}{2}(x^2+1)^{-1}e^{x^2} + C\right\} = -\frac{1}{2}(x^2+1)^{-2}\cdot 2xe^{x^2} + \frac{1}{2}(x^2+1)^{-1}\cdot 2xe^{x^2}$

$$= \frac{-xe^{x^2}}{(x^2+1)^2} + \frac{xe^{x^2}}{(x^2+1)} = \frac{xe^{x^2}}{(x^2+1)}\left\{-\frac{1}{(x^2+1)} + 1\right\}$$

$$= \frac{xe^{x^2}}{(x^2+1)}\left\{\frac{\cancel{-1} + x^2 \cancel{+1}}{(x^2+1)}\right\} = \frac{x^3e^{x^2}}{(x^2+1)^2}$$

Method 2: $u = x^2 e^{x^2}$

$\frac{du}{dx} = 2xe^{x^2} + 2x^3e^{x^2} \Rightarrow du = 2xe^{x^2}(x^2+1)$

$dv = \frac{x\,dx}{(x^2+1)^2} \Rightarrow v = -\frac{1}{2}(x^2+1)^{-1}$

$$\therefore \int \frac{x^3 e^{x^2}}{(x^2+1)^2}dx = -\frac{x^2 e^{x^2}}{2(x^2+1)} + \int (\cancel{x^2+1})^{-1} \cdot x e^{x^2} (\cancel{x^2+1})\, dx$$

$$= \frac{-x^2 e^{x^2}}{2(x^2+1)} + \int x e^{x^2} dx = e^{x^2}\left\{ -\frac{x^2}{2(x^2+1)} + \frac{1}{2} \right\} + C$$

$$= e^{x^2}\left\{ \frac{-x^2}{2(x^2+1)} + \frac{x^2+1}{2(x^2+1)} \right\} = \frac{e^{x^2}}{2(x^2+1)} + C$$

53 I.c) (5)

$$\int x^n \ln x\, dx \qquad U = \ln x \qquad dv = x^n dx$$
$$du = x^{-1} dx \qquad v = \frac{1}{(n+1)} x^{n+1}$$

$$\Rightarrow \int x^n \ln x\, dx = \frac{\ln x \cdot x^{n+1}}{(n+1)} - \frac{1}{(n+1)} \int x^{n+1} \cdot \frac{dx}{x} = \frac{\ln x \cdot x^{n+1}}{(n+1)} - \frac{1}{(n+1)} \int x^n dx$$

$$= \frac{\ln x \cdot x^{n+1}}{(n+1)} - \frac{1}{(n+1)^2} \cdot x^{n+1} + C = \frac{x^{n+1}}{(n+1)^2}\left\{ -1 + (n+1) \ln|x| \right\} + C$$

I.d) (4)
$$\int \tan^3 \frac{\pi x}{2} \sec^2 \frac{\pi x}{2}\, dx$$

$$v = \tan \frac{\pi x}{2} \quad \therefore dv = \frac{\pi}{2} \sec^2 \frac{\pi x}{2}\, dx$$

$$= \frac{2}{\pi} \int v^3 dv = \frac{2}{\pi} \cdot \frac{1}{4} v^4 + C = \frac{1}{2\pi} v^4 + C = \frac{1}{2\pi} \tan^4 \frac{\pi x}{2} + C$$

Check: $\frac{d}{dx}\left(\tan^4 \frac{\pi x}{2}\right) \cdot \frac{1}{2\pi} = \frac{1}{\cancel{2}\pi} \cdot \cancel{4} \tan^3 \frac{\pi x}{2} \cdot \frac{\cancel{\pi}}{\cancel{2}} \cdot \sec^2 \frac{\pi}{2} x = \tan^3 \frac{\pi x}{2} \sec^2 \frac{\pi x}{2}$

I.e) (5)
$$\int \sqrt{\tan x}\, \sec^4 x\, dx = \int (\tan x)^{\frac{1}{2}} \sec^2 x \cdot \sec^2 x\, dx$$

$$= \int (\tan x)^{1/2} (1 + \tan^2 x) \cdot \sec^2 x\, dx = \int (\tan x)^{1/2} \sec^2 x\, dx + \int (\tan x)^{5/2} \sec^2 x\, dx$$

$$= \int u^{1/2} du + \int u^{5/2} du = \frac{2}{3} u^{3/2} + \frac{2}{7} u^{7/2} + C = 2u^{3/2}\left[\frac{1}{3} + \frac{1}{7} u^2 \right] + C$$

$$= 2(\tan x)^{3/2}\left[\frac{1}{3} + \frac{1}{7} \tan^2 x \right] + C$$

Check: $\frac{d}{dx}\left\{ 2(\tan x)^{3/2}\left[\frac{1}{3}+\frac{1}{7}\tan^2 x\right]+C\right\}$

$$= 2\cdot\frac{3}{2}(\tan x)^{1/2}\sec^2 x\left[\frac{1}{3}+\frac{1}{7}\tan^2 x\right] + 2(\tan x)^{3/2}\left[\frac{2}{7}\tan x\cdot\sec^2 x\right]$$

$$= 3(\tan x)^{1/2}\sec^2 x\left[\frac{1}{3}+\frac{1}{7}\tan^2 x\right] + \frac{4}{7}\tan^{5/2}x\sec^2 x$$

$$= (\tan x)^{1/2}\sec^2 x + \frac{3}{7}(\tan x)^{5/2}\sec^2 x + \frac{4}{7}(\tan x)^{5/2}\sec^2 x$$

$$= (\tan x)^{1/2}\sec^2 x + (\tan x)^{5/2}\sec^2 x$$

$$= (\tan x)^{1/2}\sec^2 x\left[1+\tan^2 x\right] = (\tan x)^{1/2}\sec^2 x\cdot\sec^2 x = \underline{(\tan x)^{1/2}\sec^4 x}$$

I.f) (5) $\int \sin^n x\,dx = \int \sin^{n-2}x\sin^2 x\,dx = \int \sin^{n-2}x(1-\cos^2 x)dx$

$$= \int \sin^{n-2}x\,dx - \int \underbrace{\sin^{n-2}x\cos x\,dx}_{dV=v^{n-2}dv}\cdot\cos x \qquad u=\cos x \quad du=-\sin x\,dx$$

$$\therefore V = \frac{1}{(n-1)}v^{n-1} = \frac{1}{(n-1)}\sin^{n-1}x$$

$$= \int \sin^{n-2}x\,dx + \frac{\cos x}{(n-1)}\sin^{n-1}x + \frac{1}{(n-1)}\int \sin^{n-1}x\cdot\sin x\,dx$$

$$= \int \sin^{(n-2)}x\,dx + \frac{1}{(n-1)}\cos x\sin^{n-1}x + \frac{1}{(n-1)}\int \sin^n x\,dx$$

$$\therefore \left[1-\frac{1}{(n-1)}\right]\int \sin^n x\,dx = \frac{1}{(n-1)}\cos x\sin^{n-1}x + \cancel{\frac{1}{(n-1)}}\int \sin^{n-2}x\,dx$$

$$\frac{n}{(n-1)}\int \sin^n x\,dx = \frac{1}{(n-1)}\cos x\sin^{n-1}x + \cancel{\frac{1}{(n-1)}}\int \sin^{n-2}x\,dx$$

$$\therefore \int \sin^n x\,dx = \frac{1}{n}\cos x\sin^{n-1}x + \frac{(n-1)}{n}\int \sin^{n-2}x\,dx$$

OR

$$\int \cos^n x\,dx = \int \cos^{n-2}x\cos^2 x\,dx = \int \cos^{n-2}x(1-\sin^2 x)dx$$

$$= \int \cos^{n-2}x\,dx - \int \cos^{n-2}x\sin x\cdot\sin x\,dx$$

$U = \sin x \qquad dV = \cos^{n-2} x \sin x \, dx$

$dU = \cos x \, dx \qquad V = -\frac{1}{(n-1)} \cos^{n-1} x$

$- \int \cos^{n-2} x \sin x \cdot \sin dx$

$\therefore \int \cos^n x \, dx = \int \cos^{n-2} x \, dx + \frac{1}{(n-1)} \cos^{n-1} x \sin x - \frac{1}{(n-1)} \int \cos^n x \, dx$

$\left[1 + \frac{1}{(n-1)}\right] \int \cos^n x \, dx = + \frac{1}{(n-1)} \cos^{n-1} x \sin x + \int \cos^{n-2} x \, dx$

$\frac{n}{n-1} \int \cos^n x \, dx = + \frac{1}{(n-1)} \cos^{n-1} x \sin x + \int \cos^{n-2} x \, dx$

$\therefore \int \cos^n x \, dx = \frac{n-1}{n} \int \cos^{n-2} x \, dx + \frac{1}{n} \left[\cos^{n-1} x \sin x\right] + \cancel{c}$

II.a) (5) $\int \frac{\sqrt{4x^2+9}}{x^4} dx = 2 \int \frac{\sqrt{x^2 + 9/4}}{x^4} dx$

$x = 3/2 \tan\theta \Rightarrow dx = \frac{3}{2} \sec^2\theta \, d\theta$

$= 2 \int \frac{\sqrt{9/4 \tan^2\theta + 9/4}}{3^4/2^4 \tan^4\theta} \frac{3}{2} \sec^2\theta \, d\theta$

$= 2 \int \frac{\frac{3}{2} \sec\theta \cdot \frac{3}{2} \sec^2\theta \, d\theta}{(\frac{3}{2})^4 \tan^4\theta} = 2 \cdot \left(\frac{2}{3}\right)^2 \int \frac{\sec^3\theta \, d\theta}{\tan^4\theta}$

$= \frac{8}{9} \int \cancel{\sin} \frac{\cos^4\theta}{\sin^4\theta} \cdot \frac{1}{\cos^3\theta} d\theta = \frac{8}{9} \int (\sin\theta)^{-4} \cos\theta \, d\theta$

$= \frac{8}{9} \int v^{-4} dv = \frac{8}{9} \cdot -\frac{1}{3} v^{-3} + c = -\frac{8}{27} (\sin\theta)^{-3} + c$

$= -\left(\frac{2}{3}\right)^3 \frac{1}{(\sin\theta)^3} + c$

$x = \frac{3}{2} \tan\theta \Rightarrow \tan\theta = \frac{2x}{3} \Rightarrow \sin\theta = \frac{2x}{\sqrt{4x^2+9}}$

$= -\left(\frac{2}{3}\right)^3 \frac{(4x^2+9)^{3/2}}{8x^3} + c = -\frac{1}{27} \left(\frac{\sqrt{4x^2+9}}{x}\right)^3 + c$

$\underline{\text{Check}}$: $\frac{d}{dx}\left\{-\frac{1}{27}\left(\frac{\sqrt{4x^2+9}}{x}\right)^3\right\} = \frac{d}{dx}\left\{-\frac{1}{27}\left(4+9x^{-2}\right)^{3/2}\right\}$ (5)

$$= -\frac{1}{27}\cdot\frac{3}{2}\left(4+9x^{-2}\right)^{1/2}\cdot\left(-18x^{-3}\right)$$

$$= -\frac{1}{\cancel{9}}\cdot\left(\cancel{-9}\right)\frac{\sqrt{4+9x^{-2}}}{x^3} = \frac{\sqrt{9+4x^2}}{x^4} \checkmark$$

II. b) (5) $\displaystyle\int\frac{x\,dx}{\sqrt{x^2-6x+5}} = \int\frac{x\,dx}{\sqrt{(x-3)^2-2^2}}$

$$u = x-3 = 2\sec\theta$$

$$\therefore\ x = 2\sec\theta + 3$$

$$dx = 2\sec\theta\tan\theta\,d\theta$$

$$= \int\frac{(2\sec\theta+3)\cdot 2\sec\theta\tan\theta\,d\theta}{\sqrt{4\sec^2\theta-4}}$$

$$= \int\frac{(2\sec\theta+3)\,\cancel{2}\sec\theta\,\cancel{\tan\theta}\,d\theta}{\cancel{2}\,\cancel{\tan\theta}}$$

$$= \int(2\sec^2\theta + 3\sec\theta)\,d\theta = 2\int\sec^2\theta\,d\theta + 3\int\sec\theta\,d\theta$$

$$= 2\tan\theta + 3\ln|\sec\theta+\tan\theta| + C$$

$$\sec\theta = \frac{x-3}{2} = \frac{HYP}{ADJ} \quad \therefore\ OPP = \sqrt{(x-3)^2-2^2} = \sqrt{x^2-6x+5}$$

$$\therefore\ \tan\theta = \frac{OPP}{ADJ} = \frac{\sqrt{x^2-6x+5}}{2}$$

$$\therefore\ \int\frac{x\,dx}{\sqrt{x^2-6x+5}} = 2\cdot\frac{1}{2}\sqrt{x^2-6x+5} + 3\ln\left|\frac{x-3}{2}+\frac{1}{2}\sqrt{x^2-6x+5}\right| + C$$

$$= \sqrt{x^2-6x+5} + 3\ln\left|(x-3)+\sqrt{x^2-6x+5}\right| \underbrace{-3\ln 2 + C}_{"C}$$

$$= (x^2-6x+5)^{1/2} + 3\ln\left|(x-3)+(x^2-6x+5)^{1/2}\right| + C$$

CHECK: $\frac{d}{dx}\left\{ (x^2-6x+5)^{1/2} + 3\ln\left|(x-3)+\sqrt{x^2-6x+5}\right| \right\}$

$$= \frac{1}{2}(x^2-6x+5)^{-1/2}(2x-6) + \frac{3\left[1+\frac{1}{2}(x^2-6x+5)^{-1/2}(2x-6)\right]}{(x-3)+\sqrt{x^2-6x+5}}$$

$$= \frac{x-3}{\sqrt{x^2-6x+5}} + \frac{3\left[1+\frac{x-3}{\sqrt{x^2-6x+5}}\right]}{(x-3)+\sqrt{x^2-6x+5}}$$

$$= \frac{x-3}{\sqrt{x^2-6x+5}} + \frac{3\left[(x-3)+\sqrt{x^2-6x+5}\right]}{\sqrt{x^2-6x+5}\left[(x-3)+\sqrt{x^2-6x+5}\right]}$$

$$= \frac{(x-3)}{\sqrt{x^2-6x+5}} + \frac{3}{\sqrt{x^2-6x+5}}\left\{\cancel{\frac{(x-3)+\sqrt{x^2-6x+5}}{(x-3)+\sqrt{x^2-6x+5}}}\right\}$$

$$= \frac{x-\cancel{3}+\cancel{3}}{\sqrt{x^2-6x+5}} = \frac{x}{\sqrt{x^2-6x+5}} \quad \checkmark$$

II. c) (7) $\langle B\rangle = \frac{1}{R}\int_0^R \frac{2mL}{(r^2+L^2)^{3/2}}\,dr$

$$= \frac{2mL}{R}\int_0^R \frac{dr}{(r^2+L^2)^{3/2}} \qquad r = L\tan\theta \Rightarrow dr = L\sec^2\theta\,d\theta$$

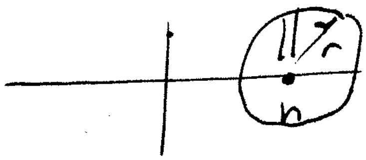
$$\theta = \arctan(r/L)$$

$$= \frac{2mL}{R}\int_{\alpha_1=\arctan(0)}^{\alpha_2=\arctan(R/L)} \frac{L\sec^2\theta\,d\theta}{(L^2\tan^2\theta+L^2)^{3/2}} = \frac{2mL}{R}\int_0^{\alpha_2}\frac{L\sec^2\theta\,d\theta}{L^3\sec^3\theta}$$

$$= \frac{2m}{RL}\int_0^{\alpha_2}\cos\theta\,d\theta = \frac{2m}{RL}\sin\theta\Big|_0^{\alpha_2} = \frac{2m}{RL}\left[\sin\alpha_2 - \sin 0\right]$$

$$= \frac{2m}{RL}\sin\left(\arctan\left(R/L\right)\right) = \frac{2m}{RL}\cdot\frac{R}{\sqrt{R^2+L^2}} = \frac{2m}{L\sqrt{R^2+L^2}}$$

II. d) (8)



Method 1 : Shells

$$\frac{1}{2}V = 2\pi\int_{h-r}^{h+r}\sqrt{r^2-(x-h)^2}\;x\,dx$$

$$u = x-h \qquad dx = du \qquad x = u+h$$

$$= 2\pi\int_{-r=u(h-r)}^{r=u(h+r)}\sqrt{r^2-u^2}\,(u+h)\,du = 2\pi\int_{-r}^{r}\sqrt{r^2-u^2}\,u\,du + 2\pi h\int_{-r}^{r}\sqrt{r^2-u^2}\,du$$

$$= -\pi\cdot\underbrace{\frac{2}{3}(r^2-u^2)\Big|_{-r}^{r}}_{0} + 2\pi h\int_{-r}^{r}\sqrt{r^2-u^2}\,du$$

$$= 2\pi h\int_{-r}^{r}\sqrt{r^2-u^2}\,du \qquad u = r\sin\theta \Rightarrow du = r\cos\theta\,d\theta$$

$$\sin\theta = u/r \Rightarrow \theta = \arcsin(u/r)$$

$$= 2\pi h\int_{\arcsin(-1)}^{\arcsin(1)} \cancel{\sin}\sqrt{r^2-r^2\sin^2\theta}\;r\cos\theta\,d\theta = 2\pi h\int_{-\pi/2}^{\pi/2} r^2\cos^2\theta\,d\theta$$

$$= 2\pi r^2\cdot\frac{h}{2}\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta$$

$$= \frac{2\pi r^2}{2}h\theta\Big|_{-\pi/2}^{\pi/2} + \pi r^2\cdot\frac{h}{2}\sin 2\theta\Big|_{-\pi/2}^{\pi/2} = h\pi r^2[\pi] = h(\pi r)^2$$

$$\therefore\ Vol = 2\cdot\frac{1}{2}V = 2h(\pi r)^2$$

Method 2 : Washers $\quad \frac{1}{2}V = \pi \int_0 \{ x_2^2(y) - x_1^2(y) \} dy$

$$(x-h)^2 + y^2 = r^2 \Rightarrow (x-h)^2 = r^2 - y^2$$

$$\therefore (x-h)_{1,2} = \pm \sqrt{r^2-y^2}$$

$$\therefore x_{1,2} = h \pm \sqrt{r^2-y^2}$$

$$\therefore x_2^2(y) - x_1^2(y) = (h+\sqrt{r^2-y^2})^2 - (h-\sqrt{r^2-y^2})^2$$

$$= \cancel{h^2} + 2h\sqrt{r^2-y^2} + \cancel{(r^2-y^2)} - [\cancel{h^2} - 2h\sqrt{r^2-y^2} + \cancel{(r^2-y^2)}]$$

$$= 4h\sqrt{r^2-y^2}$$

$$\therefore \frac{1}{2}V = 4\pi h \int_0^r \sqrt{r^2-y^2}\, dy$$

$$y = r\sin\theta \quad dy = r\cos\theta\, d\theta$$

$$= 4\pi h \int_0^{\pi/2 = \arcsin(r/r)} r^2 \cos^2\theta\, d\theta$$

$$= 2\pi h r^2 \int_0^{\pi/2} (1 + \cos 2\theta) d\theta = r^2 2\pi h \theta \Big|_0^{\pi/2} + 2\frac{\pi h}{2} r^2 \sin 2\theta \Big|_0^{\pi/2}$$

$$= 2\pi h r^2 \cdot \frac{\pi}{2} + 0 = h\pi^2 r^2$$

$$\therefore V = 2 \cdot \frac{1}{2}V = 2h(\pi r)^2$$

III. a) (3) $\displaystyle\int \frac{x^3 - x + 3}{x^2 + x - 2}\,dx = \int \left[(x-1) + \frac{2x+1}{x^2+x-2} \right] dx$

$$
\begin{array}{r|l}
 & x - 1 \\
\hline
x^2 + x - 2 & x^3 - x + 3 \\
 & x^3 + x^2 - 2x \\
\hline
 & -x^2 + x + 3 \\
 & -x^2 - x + 2 \\
\hline
 & 2x + 1
\end{array}
$$

$$= \frac{1}{2}x^2 - x + \int \frac{2x+1}{(x+2)(x-1)}\,dx$$

$$\frac{2x+1}{(x+2)(x-1)} = \frac{A_1}{(x+2)} + \frac{A_2}{(x-1)} \Rightarrow 2x+1 = A_1(x-1) + A_2(x+2)$$

$$x = 1 \Rightarrow 3 = 3A_2 \Rightarrow A_2 = 1$$

$$x = -2 \Rightarrow -3 = -3A_1 \Rightarrow A_1 = 1$$

$$\therefore \frac{2x+1}{(x+2)(x-1)} = \frac{1}{(x+2)} + \frac{1}{(x-1)} = \frac{(x-1)+(x+2)}{(x+2)(x-1)} = \frac{2x+1}{(x+2)(x-1)}$$

check ✓ ✓

$$= \frac{1}{2}x^2 - x + \int \left[\frac{1}{(x+2)} + \frac{1}{(x-1)} \right] dx = \frac{1}{2}x^2 - x + \ln|x+2| + \ln|x-1| + C$$

Check: $\displaystyle\frac{d}{dx}\left\{ \frac{1}{2}x^2 - x + \ln|x+2| + \ln|x-1| \right\} = x - 1 + \frac{1}{(x+2)} + \frac{1}{(x-1)}$

$$= (x-1) + \frac{2x+1}{x^2+x-2} \quad ✓$$

III. b) (5) $\displaystyle\int \frac{2x-3}{(x-1)^2}\,dx$

$$\frac{2x-3}{(x-1)^2} = \frac{A_1}{(x-1)} + \frac{A_2}{(x-1)^2} \Rightarrow 2x - 3 = A_1(x-1) + A_2$$

$$x = 1 \Rightarrow \boxed{-1 = A_2}$$

$$2x - 3 = A_1(x-1) - 1 \Rightarrow \boxed{A_1 = 2}$$

$$\therefore \frac{2x-3}{(x-1)^2} = \frac{2}{(x-1)} - \frac{1}{(x-1)^2} = \frac{2(x-1)-1}{(x-1)^2} \quad \checkmark \text{ Check}$$

$$\therefore \int \frac{2x-3}{(x-1)^2}dx = \int \frac{2}{(x-1)}dx - \int \frac{dx}{(x-1)^2} = 2\ln|x-1| + (x-1)^{-1} + C$$

Check : $\frac{d}{dx}\left\{ 2\ln|x-1| + \frac{1}{(x-1)} \right\} = \frac{2}{(x-1)} - \frac{1}{(x-1)^2} = \frac{2x-3}{(x-1)^2}$ ✓

III. c) (7) $\int \frac{6x^2+1}{x^2(x-1)^3}dx$

$$\frac{6x^2+1}{x^2(x-1)^3} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{(x-1)} + \frac{A_4}{(x-1)^2} + \frac{A_5}{(x-1)^3}$$

$$6x^2+1 = A_1x(x-1)^3 + A_2(x-1)^3 + A_3x^2(x-1)^2 + A_4x^2(x-1) + A_5x^2$$

$$x=0 \Rightarrow 1 = -A_2 \Rightarrow \boxed{A_2=-1}$$

$$x=1 \Rightarrow 7 = A_5 \Rightarrow \boxed{A_5=7}$$

$$6x^2+1 = A_1x(x-1)^3 - (x-1)^3 + A_3x^2(x-1)^2 + A_4x^2(x-1) + 7x^2$$

$$x^4: \; 0 = A_1 + A_3 \Rightarrow \boxed{A_1 = -A_3}$$

$$6x^2+1 = A_1\left[x(x-1)^3 - x^2(x-1)^2\right] - (x-1)^3 + A_4x^2(x-1) + 7x^2$$

$$= A_1x(x-1)^2\left[(x-1)-x\right] - (x-1)^3 + A_4x^2(x-1) + 7x^2$$

$$6x^2+1 = -A_1x(x-1)^2 - (x-1)^3 + A_4x^2(x-1) + 7x^2$$

$$x^3: \; 0 = -A_1 - 1 + A_4 \Rightarrow A_4 = (A_1+1)$$

$$6x^2+1 = -A_1x(x-1)^2 - (x-1)^3 + (A_1+1)x^2(x-1) + 7x^2$$

$$= A_1\left[x^2(x-1) - x(x-1)^2\right] - (x-1)^3 + x^2(x-1) + 7x^2$$

$$6x^2+1 = A_1 x(x-1)[x-(x-1)] - (x-1)^3 + x^2(x-1) + 7x^2$$

$$6x^2+1 = A_1 x(x-1) - \underbrace{(x-1)^3}_{x^3-3x^2+3x-1} + x^2(x-1) + 7x^2$$

$$x^1: \; 0 = -A_1 - 3$$

$$\therefore \boxed{A_1 = -3} \qquad \therefore \boxed{A_3 = 3} \qquad \boxed{A_4 = -2}$$

$$\therefore \frac{6x^2+1}{x^2(x-1)^3} = \frac{-3}{x} - \frac{1}{x^2} + \frac{3}{(x-1)} - \frac{2}{(x-1)^2} + \frac{7}{(x-1)^3}$$

Check: $\frac{6x^2+1}{x^2(x-1)^3} = \frac{-3x(x-1)^3 - (x-1)^3 + 3x^2(x-1)^2 - 2x^2(x-1) + 7x^2}{x^2(x-1)^3}$

$$= (x-1)^3[-3x-1]$$

$$= \frac{\cancel{-3x^4} + \cancel{9x^3} - \cancel{9x^2} + \cancel{3x} - \cancel{x^3} + \overbrace{3x^2 - \cancel{3x} + 1 + \cancel{3x^4} - \cancel{6x^3} + 3x^2}^{=6x} - \cancel{3x^3} + \cancel{2x^2} + \cancel{7x^2}}{x^2(x-1)^3}$$

$$= \frac{6x+1}{x^2(x-1)^3} \checkmark$$

$$\therefore \int \frac{6x^2+1}{x^2(x-1)^3}\,dx = \int \left\{ -\frac{3}{x} - \frac{1}{x^2} + \frac{3}{(x-1)} - \frac{2}{(x-1)^2} + \frac{7}{(x-1)^3} \right\} dx$$

$$= -3\ln|x| + \frac{1}{x} + 3\ln|x-1| + \frac{2}{(x-1)} + -\frac{7}{2}(x-1)^{-2} + C$$

$$= 3\ln\left|\frac{x}{x-1}\right| + \frac{1}{x} + (x-1)^{-1}\left[2 - \frac{7}{2(x-1)}\right] + C$$

III.d) (5) $\int_1^{5} \frac{x-1}{x^2(x+1)}$

12

$$\frac{x-1}{x^2(x+1)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{(x+1)}$$

$$x-1 = A_1 x(x+1) + A_2(x+1) + A_3 x^2$$

$$x=0 \Rightarrow \boxed{-1 = A_2}$$

$$x=-1 \Rightarrow \boxed{-2 = A_3}$$

$$x-1 = A_1 x(x+1) - (x+1) - 2x^2$$

$$x^2:\ 0 = A_1 - 2 \Rightarrow \boxed{A_1 = 2}$$

$$\therefore \frac{x-1}{x^2(x+1)} = \frac{2}{x} + \frac{-1}{x^2} - \frac{2}{(x+1)}$$

<u>Check:</u> $= \frac{2x(x+1) - (x+1) - 2x^2}{x^2(x+1)} = \frac{\cancel{2x^2} + \overbrace{2x - x}^{x} - 1 - \cancel{2x^2}}{x^2(x+1)} = \frac{x-1}{x^2(x+1)}$ ✓

$$\therefore \int_1^5 \frac{x-1}{x^2(x+1)}\,dx = \int_1^5 \left\{ \frac{2}{x} - \frac{1}{x^2} - \frac{2}{(x+1)} \right\} dx = \left\{ 2\ln|x| + \frac{1}{x} - 2\ln|x+1| \right\} \Big|_1^5$$

$$= \left\{ \ln\left[\frac{x}{x+1}\right]^2 + \frac{1}{x} \right\} \Big|_1^5 = \ln\frac{5}{6} + \frac{1}{5} - \ln\frac{1}{2} - 1$$

$$= \ln 5 - \ln 6 + \frac{4}{5} - \ln 2$$

III.e) (5) $\int \frac{dx}{x^2(a+bx)}$

$$\frac{1}{x^2(a+bx)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{(a+bx)}$$

$$1 = A_1 x(a+bx) + A_2(a+bx) + A_3 x^2$$

$x=0 \Rightarrow 1 = A_2 a \Rightarrow A_2 = 1/a \qquad x = -a/b \Rightarrow 1 = A_3 a^2/b^2 \Rightarrow A_3 = b^2/a^2$ (13)

$\therefore\ 1 = A_1 x(a+bx) + \frac{1}{a}(a+bx) + \frac{b^2}{a^2}x^2 \Rightarrow x^1:\ 0 = A_1 a + \frac{b}{a} \Rightarrow A_1 = \frac{-b}{a^2}$

$$\therefore\ \frac{1}{x^2(a+bx)} = \frac{-b}{a^2 x} + \frac{1}{ax^2} + \frac{b^2}{a^2(a+bx)}$$

Check:

$$\frac{-b/a^2\, x(a+bx) + 1/a\,(a+bx) + b^2/a^2 \cdot x^2}{x^2(a+bx)}$$

$$= \frac{\cancel{-b/a\, x} \cancel{- b^2/a^2\, x^2} + 1 \cancel{+ b/a\, x} \cancel{+ b^2/a^2\, x^2}}{x^2(a+bx)} = \frac{1}{x^2(a+bx)}$$

$$\therefore \int \frac{dx}{x(a+bx)} = -\frac{b}{a^2}\int\frac{dx}{x} + \frac{1}{a}\int\frac{dx}{x^2} + \frac{b^2}{a^2}\int\frac{dx}{(a+bx)}$$

$$= \frac{-b}{a^2}\ln|x| - \frac{1}{ax} + \frac{b^2}{a^2}\cdot\frac{1}{b}\ln|a+bx|$$

$$= -\frac{b}{a^2}\ln|x| + \frac{b}{a^2}\ln|a+bx| - \frac{1}{ax} + C$$

$$= \frac{b}{a^2}\ln\left|\frac{a+bx}{x}\right| - \frac{1}{ax} + C$$

$$= -\frac{b}{a^2}\ln\left|\frac{x}{a+bx}\right| - \frac{1}{ax} + C$$

IV a) (20) $\int \frac{e^x}{1-\tan e^x}dx = \int \frac{du}{1-\tan u}$ (71.)

$(u = e^x)$

(21) $\int \frac{\cos x}{1+\sin^2 x}dx \to u = \sin x \to \int \frac{du}{1+u^2}$ (23.)

22) $\int \frac{1/t \, dt}{[1+(\ln t)^2]} \to \begin{matrix} u = \ln t \\ du = dt/t \end{matrix} \to \int \frac{du}{1+u^2}$ (23.)

23) $\int \frac{1}{1+e^{2x}}dx \to \begin{matrix} u = e^x \\ du = e^x dx \end{matrix} \longrightarrow \int \frac{du}{u(1+u^2)}$

$\hookrightarrow u = 2x \quad du = 2dx \to \frac{1}{2}\int \frac{du}{1+e^u}$ (84.)

24) $\int \frac{dx}{\sqrt{x}(1+\sqrt{x})} \quad \begin{matrix} u = x^{1/2} \\ du = \frac{1}{2}x^{-1/2}dx \end{matrix} \longrightarrow 2\int \frac{du}{(1+u)} \to \int \frac{dw}{w}$ (2.)

$\hookrightarrow \int \frac{}{}$

25) $\int \frac{\cos t \, dt}{3+2\sin t+\sin^2 t} = \int \frac{du}{u^2+2u+3}$ (14.)

26) $\int x^2\sqrt{2+9x^2}\,dx \quad \begin{matrix} u = 3x \\ a = \sqrt{2} \end{matrix} \to \frac{1}{27}\int u^2\sqrt{a^2+u^2}$ (27.)

27) $\int \frac{dx}{x^2\sqrt{2+9x^2}} \quad \begin{matrix} u = 3x \\ a = \sqrt{2} \end{matrix} \to 9\int \frac{du}{u^2\sqrt{a^2+u^2}}$ (35.)

28) $\int \sqrt{3+x^2}\,dx \quad \begin{matrix} a = \sqrt{3} \\ u = x \end{matrix} \to \int \sqrt{a^2+u^2}\,du$ (26.)

29) $\int e^x\sqrt{1+e^{2x}}\,dx \quad \begin{matrix} u = e^x \\ du = e^x dx \end{matrix} \longrightarrow \int \sqrt{1+u^2}\,du$ (26.)

30) $\int \frac{dx}{\sqrt{x}(x-4)^{3/2}} \quad \begin{matrix} u = \sqrt{x} \\ du = \frac{1}{2}\frac{dx}{\sqrt{x}} \end{matrix} \to 2\int \frac{du}{(u^2-a^2)^{3/2}}$ (36.)

$a = 2$

31) $\int \sin^4 2x\,dx = \frac{1}{2}\int \sin^4 u\,du$ 50.

32) $\int \frac{\cos^3\sqrt{x}}{\sqrt{x}}\,dx \to u = x^{1/2} \to 2\int \cos^3 u\,du$ 51.

$du = \frac{1}{2}x^{-1/2}dx$

33) $\int \frac{dx}{\sqrt{x}(1-\cos\sqrt{x})} \to u = x^{1/2} \to 2\int \frac{du}{1-\cos u}$ 57.

$du = \frac{1}{2}x^{-1/2}$

34) $\int \frac{dx}{1-\tan 5x} \to \frac{1}{5}\int \frac{du}{1-\tan u}$ 71.

35) $\int t^4 \cos t\,dt$ 55.

36) $\int \sqrt{x}\arctan((\sqrt{x})^3)\,dx \longrightarrow \frac{2}{3}\int \arctan u\,du$ 77.

$u = x^{3/2}$

$du = \frac{3}{2}x^{1/2}dx \Rightarrow \sqrt{x}\,dx = \frac{2}{3}du$

37) $\int x\,\mathrm{arcsec}(x^2+1)\,dx \to \frac{1}{2}\int \mathrm{arcsec}\,u\,du$ 79.

$u = x^2+1$

$du = 2x\,dx$

38) $\int (\ln x)^3 dx$ 91.

39) $\int \frac{\ln x\,dx}{x(3+2\ln x)} \to u = \ln x \to \int \frac{u\,du}{(a+bu)}$ 3.

$du = \frac{dx}{x}$

$a = 3 \quad b = 2$

40) $\int \frac{e^x\,dx}{(1-e^{2x})^{3/2}} \quad u = e^x \quad du = e^x dx \quad \int \frac{du}{(1-u^2)^{3/2}}$ 45.

$a = 1$

41) $\int \frac{\sqrt{2-2x-x^2}}{x+1}\,dx = \int \frac{\sqrt{-(x+1)^2+3}}{(x+1)} \xrightarrow{u = x+1} \int \frac{\sqrt{a^2-u^2}}{u}\,du$

$a = \sqrt{3}$

39.

42) $\displaystyle\int \frac{dx}{(x^2-6x+10)^2} = \int \frac{dx}{(x^2-6x+9+1)^2} = \int \frac{dx}{[(x-3)^2+1^2]}$

$u = x \cancel{+3}$

$\displaystyle = \int \frac{du}{u^2+1^2}$ 23.

43) $\displaystyle\int \frac{x\,dx}{x^4-6x^2+10} = \frac{1}{2}\int \frac{du}{u^2-6u^2+10} = \frac{1}{2}\int \frac{du}{(u-3)^2+1^2}$

$u = x^2$
$du = 2x\,dx$

x

$\displaystyle = \frac{1}{2}\int \frac{dw}{w^2+1}$ 23.

44) $\displaystyle\int (2x-3)^2\sqrt{(2x-3)^2+4}$ $\quad u = 2x-3$, $du = 2dx$ $\displaystyle = \frac{1}{2}\int u^2\sqrt{u^2+2^2}$ 21.

45) $\displaystyle\int \frac{x\,dx}{\sqrt{x^4-6x^2+5}} = \frac{1}{2}\int \frac{du}{\sqrt{u^2-6u+5}} = \frac{1}{2}\int \frac{du}{\sqrt{(u-3)^2-4}}$

$\displaystyle = \frac{1}{2}\int \frac{dw}{\sqrt{w^2-2^2}}$ 31.

46) $\displaystyle\int \frac{\cos x\,dx}{\sqrt{\sin^2 x+1}} \rightarrow$ $u = \sin x$, $du = \cos x\,dx$ $\displaystyle \longrightarrow \int \frac{du}{\sqrt{u^2+1}}$ 31.

47) $\displaystyle\int \frac{x^3}{\sqrt{4-x^2}}\,dx = \int \frac{u^3\,du}{\sqrt{2^2-u^2}}$

dV 43.

$\displaystyle = \int \left(\frac{u^2\,dx}{\sqrt{2^2-u^2}}\right)\cdot u$ V F

OR (Method 2): $w^2 = 4-u^2$

48) $\int \sqrt{\frac{3-x}{3+x}}\,dx = \int \sqrt{\frac{(3-x)^2}{(3+x)(3-x)}} = \int \frac{3-x}{\sqrt{9-x^2}}$

$= \int \frac{3}{\sqrt{9-x^2}} - \int \frac{x}{\sqrt{9-x^2}}$

41.

can be done with a simple u-subst.
(Don't need table)

49) $\int \frac{1}{x^{3/2}\sqrt{1-x}}\,dx$ $\qquad u = x^{1/2}$
$\qquad du = \frac{1}{2}x^{-1/2}dx$

$= 2\int \frac{du}{u^2\sqrt{u^2-u^2}}$ 44.

50) $\int x\sqrt{x^2+2x}\,dx = \int x\sqrt{x^2+2x+1-1} = \int x\sqrt{(x+1)^2-1^2}$

$= \int (u-1)\sqrt{u^2-1^2} = \int u\sqrt{u^2-1}\,du - \int \sqrt{u^2-1}\,du$ 37.

No need of Table
(Simple u-subst.)

51) $\int \frac{e^{2x}\,dx}{(1+e^x)^3} = \int \frac{e^x \cdot e^x dx}{(1+e^x)^3} = \int \frac{u \cdot du}{(1+u)^3}$ 9.

52) $\int \sec^3\theta\, d\theta$ 69.

IV.5(5)

$$\int \frac{u^2}{(a+bu)^2}\,du$$

$$a^2+2abu+b^2u^2 \overline{\big)\, u^2} \quad \frac{1}{b^2}$$

$$u^2 + \frac{2a}{b}u + \frac{a^2}{b^2}$$

$$-\frac{2a}{b}u - \frac{a^2}{b^2}$$

$$\frac{u^2}{(a+bu)^2} = \frac{1}{b^2} - \frac{\left(\frac{2a}{b}u + \frac{a^2}{b^2}\right)}{(a+bu)^2}$$

$$= \frac{1}{b^2} - \frac{a}{b}\left[\frac{2u + a/b}{(a+bu)^2}\right]$$

$$2u + a/b = A_1(a+bu) + A_2$$

$$u = -a/b$$

$$-a/b = A_2$$

$$2u + a/b = A_1(a+bu) - a/b$$

$$\boxed{A_1 = \frac{2}{b}}$$

$$\therefore \int \frac{u^2}{(a+bu)^2}\,du = \int \left\{ \frac{1}{b^2} - \frac{a}{b}\left[\frac{2/b}{(a+bu)} - \frac{a/b}{(a+bu)^2}\right]\right\} du$$

$$= \frac{1}{b^2}u - \frac{2a}{b^2}\int \frac{du}{(a+bu)} + \frac{a^2}{b^2}\int \frac{du}{(a+bu)^2}$$

$$= \frac{1}{b^2}u - \frac{2a}{b^3}\ln|a+bu| - \frac{a^2}{b^3}(a+bu)^{-1} = \frac{1}{b^3}\left(bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$$

70

IV. c) (5)

$$\int_0^5 \frac{500x\,dx}{\sqrt{26-x^2}} = 500\int_0^5 \frac{x\,dx}{\sqrt{26-x^2}} = -250\int_{26}^{1} \frac{du}{u^{1/2}} = +250\int_1^{25} u^{-1/2}\,du$$

$u = 26 - x^2$

$du = -2x\,dx$

$$= 250 \cdot 2u^{1/2}\Big|_1^{25} = 500\{\sqrt{25} - 1\} = 500 \cdot 4 = 2000 \text{ ft lbs.}$$