MATH 262

ASSIGNMENT 3 KEY

II a) $x(\theta) = 4\cos^2\theta \qquad y(\theta) = 2\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos\theta}{8\cos\theta(-\sin\theta)} = -\frac{1}{4}\csc\theta$$

• PTS OF VERTICAL TANG.: $\frac{dx}{dy} = 0 \Rightarrow dy/d\theta \neq 0 \;\&\; dx/d\theta = 0 \Rightarrow \underset{0}{\underset{\shortparallel}{\sin\theta}} \Rightarrow \theta_n = n\pi$

• PTS OF HORIZ TANG.: $\frac{dy}{dx} = 0 \Rightarrow dy/d\theta = 0 \;\&\; dx/d\theta \neq 0 \Rightarrow$ NONE

$\therefore$ PTS OF VERTICAL TANGENCY <u>ONLY</u> : $(x_n, y_n) = (4\cos^2\theta_n, 2\sin\theta_n) = (4, 0)$

II b) $L = \int_a^b ds$ For all $\theta \in [a,b)$ where graph doesn't intersect itself

Find all θ such that $x(\theta) = y(\theta) \Rightarrow a\cos^3\theta = a\sin^3\theta \Rightarrow \theta_1 = \pi/4$

$$\therefore \quad L = \int_0^{2\pi} ds = 8\int_0^{\pi/4} ds(\theta) \qquad ds(\theta) = \sqrt{(x'(\theta))^2 + (y'(\theta))^2}$$

$$= \sqrt{(3a\sin^2\theta\cos\theta)^2 + (3a\cos^2\theta - \sin\theta)^2}$$

$$= 3a\sin\theta\cos\theta\sqrt{\sin^2\theta + \cos^2\theta}$$

$$= \frac{3}{2}a\sin 2\theta$$

$$\therefore \quad L = 8\int_0^{\pi/4} \frac{3}{2}a\sin 2\theta = -12a\cdot\frac{1}{2}\cos 2\theta\Big|_0^{\pi/4}$$

$$= -6a\cos 2\theta\Big|_0^{\pi/4}$$

$$= -6a(0-1) = \underline{6a}$$

II. c) $x(\theta) = a\cos\theta \qquad y(\theta) = b\sin\theta$

About x : $$S = 2\cdot 2\pi\int_0^{\pi/2} y(\theta)\sqrt{(x'(\theta))^2 + (y'(\theta))^2}\,d\theta$$

$$= 4\pi b\int_0^{\pi/2} \sin\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,d\theta$$

$$= 4\pi ab\int_0^{\pi/2} \sin\theta\sqrt{\sin^2\theta + \frac{b^2}{a^2}\cos^2\theta}\,d\theta$$

$$= 4\pi ab\int_0^{\pi/2} \sin\theta\sqrt{1 + \frac{b^2-a^2}{a^2}\cos^2\theta}\,d\theta$$

$$= 4\pi ab \int_0^{\pi/2} \sin\theta \sqrt{1 - e^2\cos^2\theta}\, d\theta$$

Let $u = e\cos\theta$ $\therefore$ $du = -e\sin\theta\, d\theta \Rightarrow \sin\theta\, d\theta = -\frac{1}{e}du$

$$= \frac{4\pi ab}{e}\int_e^0 -du\sqrt{1-u^2} = \frac{4\pi ab}{e}\int_0^e \sqrt{1-u^2}\, du$$

$u = \sin\phi$ $\quad du = \cos\phi\, d\phi$

$$= \frac{4\pi ab}{e}\int_0^{\arcsin(e)}\sqrt{1-\sin^2\phi}\cos\phi\, d\phi = \frac{4\pi ab}{e}\int_0^{\arcsin(e)}\cos^2\phi\, d\phi$$

$$= \frac{2\pi ab}{e}\int_0^{\arcsin(e)}[1+\cos 2\phi]\, d\phi = \frac{2\pi ab}{e}\left\{\phi + \frac{1}{2}\sin 2\phi\right\}\Big|_0^{\arcsin e}$$

$$= \frac{2\pi ab}{e}\left\{\arcsin(e) + \sin(\arcsin e)\cos(\arcsin e)\right\}$$

$$= \frac{2\pi ab}{e}\left\{\arcsin(e) + e\sqrt{1-e^2}\right\}$$

About y: $2\cdot 2\pi \int_0^{\pi/2} x(\theta)\sqrt{x'(\theta)^2+y'(\theta)^2}\, d\theta = 4\pi a\int_0^{\pi/2}\cos\theta\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$

$$= 4\pi ab\int_0^{\pi/2}\cos\theta\sqrt{\frac{a^2}{b^2}\sin^2\theta+\cos^2\theta} = 4\pi ab\int_0^{\pi/2}\cos\theta\sqrt{\frac{a^2-b^2}{b^2}\sin^2\theta+1}$$

$$= 4\pi ab\int_0^{\pi/2}\cos\theta\sqrt{1+e^2\sin^2\theta}\, d\theta$$ $\quad u = e\sin\theta$ $\quad du = e\cos\theta\, d\theta \Rightarrow \cos\theta\, d\theta = \frac{1}{e}du$

$$= \frac{4\pi ab}{e}\int_0^e du\sqrt{1+u^2}$$ $\quad u = \tan\theta$, $du = \sec^2\theta\, d\theta$ $\Rightarrow \frac{4\pi ab}{e}\int_0^{\arctan e}\sec^3\theta\, d\theta$

$$= \frac{4\pi ab}{e}\int_0^{\arctan e}\sec\theta(1+\tan^2\theta)\, d\theta = \frac{4\pi ab}{e}\int_0^{\arctan e}[\sec\theta + \tan^2\theta\sec\theta]\, d\theta$$

$$= \frac{4\pi ab}{e}\left\{\frac{\sec\theta\tan\theta}{2}\Big|_0^{\arctan e} + \frac{1}{2}\int_0^{\arctan e}\sec\theta\, d\theta\right\} = \frac{4\pi ab}{2e}\left\{e\sec(\arctan e) + \ln|\sec\theta+\tan\theta|\Big|_0^{\arctan e}\right\}$$

$$= \frac{4\pi ab}{2e}\left\{e\sqrt{1+e^2} + \ln\left|\sqrt{1+e^2}+e\right|\right\}$$

III. d) $\vec{r}''(t) = -4\cos t\,\hat{i} - 3\sin t\,\hat{j}$

$\vec{r}'(0) = 3\hat{j} \quad \vec{r}(0) = 4\hat{i}$

$\vec{r}'(t) = \int \vec{r}''(t)dt = [-4\sin t + c_1]\hat{i} + [3\cos t + c_2]\hat{j} \Rightarrow \vec{r}'(0) = 3\hat{j} = (0 + c_1)\hat{i} + (3 + c_2)\hat{j}$

$\therefore \boxed{c_1 = c_2 = 0}$

$\vec{r}(t) = \int \vec{r}'(t)dt = [4\cos t + c_1]\hat{i} + [3\sin t + c_2]\hat{j} \Rightarrow \vec{r}(0) = 4\hat{i} = (4 + c_1)\hat{i} + c_2\hat{j}$

$\therefore \boxed{c_1 = c_2 = 0}$

$\therefore \vec{r}(t) = 4\cos t\,\hat{i} + 3\sin t\,\hat{j}$

IV. a) $x - 3y + 6z = 4 \Rightarrow n_1 = \hat{i} - 3\hat{j} + 6\hat{k}$

$5x + y - z = 4 \Rightarrow n_2 = 5\hat{i} + \hat{j} - \hat{k}$

$\therefore \vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 6 \\ 5 & 1 & -1 \end{vmatrix}$

$= \hat{i}(-3) - \hat{j}(-31) + \hat{k}(16) = -3\hat{i} + 31\hat{j} + 16\hat{k}$

To find a pt in common, solve 2x2 system by setting one of the above variables = 0

$\therefore$ set $z = 0$ (chosen arbitrarily) $\Rightarrow$

$$\begin{array}{r} x - 3y = 4 \\ 3(5x + y = 4) \\ \hline 16x = 16 \Rightarrow x = 1 \quad y = -1 \end{array}$$

$\therefore \vec{r}_0 = \hat{i} - \hat{j} + 0\hat{k}$

$\therefore l: \vec{r}(t) = \hat{i} - \hat{j} + t(-3\hat{i} + 31\hat{j} + 16\hat{k})$

$= (1 - 3t)\hat{i} + (-1 + 31t)\hat{j} + 16t\,\hat{k}$

$x(t) = 1 - 3t \Rightarrow x + 3/16\, z = 1$

$y(t) = -1 + 31t \Rightarrow y - 31/16\, z = -1$

$z(t) = 16t \Rightarrow t = z/16$

IV. b) $z = f(x,y) = \dfrac{xy}{x - y}$

$\dfrac{\partial}{\partial x} z = \dfrac{\partial}{\partial x}\left[\dfrac{xy}{x-y}\right] = \dfrac{(x-y)y - xy}{(x-y)^2} = \dfrac{-y^2}{(x-y)^2}$

$\dfrac{\partial z}{\partial y} = \dfrac{\partial}{\partial y}\left[\dfrac{xy}{x-y}\right] = \dfrac{(x-y)x - xy}{(x-y)^2} = \dfrac{x^2}{(x-y)^2}$

$\partial_{xx} z = \dfrac{\partial^2}{\partial x^2} z = \dfrac{\partial}{\partial x}\left[\dfrac{-y^2}{(x-y)^2}\right] = \dfrac{2y^2}{(x-y)^3} \qquad \partial_{yy} z = \dfrac{\partial}{\partial y}\left[\dfrac{x^2}{(x-y)^2}\right] = \dfrac{+2x^2}{(x-y)^3}$

$\dfrac{\partial z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left[\dfrac{x^2}{(x-y)^2}\right] = \dfrac{2x(x-y)^2 - x^2 2(x-y)}{(x-y)^4} = \dfrac{2x(x-y) - 2x^2}{(x-y)^3} = \dfrac{-2xy}{(x-y)^3} = \dfrac{\partial z}{\partial y \partial x}$

III.a) $r = 2\sin 3\theta = 2[\sin\theta\cos 2\theta + \cos 2\theta \sin 2\theta] = 2[\sin\theta(\cos^2\theta - \sin^2\theta) + \cos\theta(2\sin\theta\cos\theta)]$

$= 2[\cos^2\theta\sin\theta - \sin^3\theta + 2\sin\theta\cos^2\theta] = 2[3\cos^2\theta\sin\theta - \sin^3\theta] = 6\cos^2\theta\sin\theta - 2\sin^3\theta$

$\therefore\ r^4 = 6(r\cos\theta)^2(r\sin\theta) - 2(r\sin\theta)^3 \Rightarrow (x^2+y^2)^2 = 6x^2y - 2y^3$

III.b) $y^2 - 8x + 16 = 0 \Rightarrow r^2\sin^2\theta - 8r\cos\theta + 16 = 0 \Rightarrow r_{1,2} = \dfrac{8\cos\theta \pm \sqrt{64\cos^2\theta + 64\sin^2}}{2\sin\theta}$

$= \dfrac{8[\cos\theta \pm \sqrt{\sin^2 + \cos^2\theta}]}{2\sin\theta}$

$= \cancel{4[\cot\theta \pm \sqrt{\cot^2\theta + \csc^2\theta}]}$

$= \cancel{4[\cot\theta \pm \sqrt{1 + 2\cot^2\theta}]}$

$= \dfrac{4[\cos\theta \pm 1]}{\sin\theta} = 4[\cot\theta \pm \csc\theta]$

III.c) $r = a\sin\theta\cos^2\theta$

$x = r\cos\theta = a\sin\theta\cos^3\theta$
$y = r\sin\theta = a\sin^2\theta\cos^2\theta$ $\Rightarrow \dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$

Horizontal tangency $\Rightarrow y'(\theta) = 0$ & $x'(\theta) \neq 0$

$y'(\theta) = \dfrac{d}{d\theta}[a\sin^2\theta\cos^2\theta] = 2a\sin\theta\cos^3\theta - 2a\cos\theta\sin^3\theta$

$x'(\theta) = \dfrac{d}{d\theta}[a\sin\theta\cos^3\theta] = a\cos^4\theta + 3a\cos^2\theta\sin^2\theta$

$y'(\theta) = 0 = \sin\theta\cos^3\theta - \cos\theta\sin^3\theta = 0 \Rightarrow \sin\theta\cos\theta(\cos^2\theta - \sin^2\theta) = 0$

$\Rightarrow \frac{1}{2}\sin 2\theta\cos 2\theta = 0$

$\theta_{n,1} = \dfrac{n\pi}{2} \qquad \theta_{n,2} = \dfrac{n\pi}{4}$

$x'(\theta) = a\cos^2\theta(\cos^2\theta - 3\sin^2\theta) \Rightarrow$ Rules out $\boxed{\theta_{n,2}}$

III.d) $r = a(1+\cos\theta) \qquad 0 \leq \theta \leq \pi$ (Exercise 46, actually. sorry ☺)

$S = 2\pi\int_0^\pi r(\theta)\sin\theta\sqrt{r^2(\theta) + [r'(\theta)]^2}\,d\theta \qquad r'(\theta) = -a\sin\theta$

$= 2\pi a^2\int_0^\pi (1+\cos\theta)\sin\theta\sqrt{(1+\cos\theta)^2 + \sin^2\theta}\,d\theta = 2\pi a^2\int_0^\pi (1+\cos\theta)\sin\theta\sqrt{1+2\cos\theta+\sin^2\theta+\cos^2\theta}\,d\theta$

$= 2\pi a^2\int_0^\pi \sin(1+\cos\theta)\sqrt{2(1+\cos\theta)}\,d\theta = 2\sqrt{2}\pi a^2\int_0^\pi (1+\cos\theta)^{3/2}\sin\theta\,d\theta$

$u = 1+\cos\theta \quad du = -\sin\theta\,d\theta \Rightarrow -2\sqrt{2}\pi a^2\int_2^0 u^{3/2}\,du = 2\sqrt{2}\pi a^2\cdot\frac{2}{5}u^{5/2}\Big|_0^2 = \dfrac{32\pi a^2}{5}$

IV.c) $f(x,y) = 9 - x^2 - y^2$

$\nabla f = -2x\hat{i} + 2y\hat{j}$

a) $D_u f(1,2) = \nabla f(1,2)\cdot \hat{u}_1 = [-2\hat{i} - 4\hat{j}]\cdot[\cos -\pi/4\,\hat{i} + \sin -\pi/4\,\hat{j}]$

$= -2(\hat{i}+2\hat{j})\cdot\left(\frac{\sqrt{2}}{2}\hat{i} - \frac{\sqrt{2}}{2}\hat{j}\right) = -\sqrt{2}\,(\hat{i}+2\hat{j})\cdot(\hat{i}-\hat{j}) = -\sqrt{2}\,(1-2) = \sqrt{2}$

b) $D_u f(1,2) = \nabla f(1,2)\cdot\hat{u}_2 = -2(\hat{i}+2\hat{j})\cdot\left(\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right) = -2\left[\frac{1}{2} + \sqrt{3}\right] = -[1+2\sqrt{3}]$

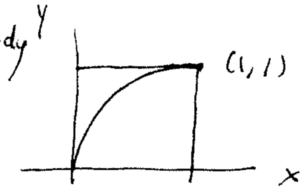
IV.d)

$$\int_0^{\pi/4}\int_0^{\cos\theta} 3r^2\sin\theta\,dr\,d\theta = 3\int_0^{\pi/4}\sin\theta\left[\int_0^{\cos\theta} r^2\,dr\right]d\theta = 3\int_0^{\pi/4}\sin\theta\cdot\frac{1}{3}r^3\Big|_0^{\cos\theta}d\theta$$

$$= \int_0^{\pi/4}\sin\theta\cdot\cos^3\theta\,d\theta \Rightarrow \begin{array}{l} u = \cos\theta \\ du = -\sin\theta\,d\theta \end{array} \Rightarrow -\int_1^{\sqrt{2}/2} u^3\,du = \int_{\sqrt{2}/2}^1 u^3\,du = \frac{1}{4}u^4\Big|_{\sqrt{2}/2}^1$$

$$= \frac{1}{4}\left[1 - \left(\sqrt{1/2}\right)^4\right] = \frac{1}{4}\left[1 - \frac{1}{4}\right] = 3/16$$

IV.e) $\displaystyle\int_0^1\int_y^{\sqrt{y}} x^2y^2\,dx\,dy$



$$\int_0^1\int_y^{\sqrt{y}} x^2y^2\,dx\,dy = \int_0^1 y^2\left\{\int_y^{\sqrt{y}} x^2\,dx\right\}dy = \int_0^1 y^2\left\{\frac{1}{3}x^3\Big|_y^{y^{1/2}}\right\}dy$$

$$= \frac{1}{3}\int_0^1 y^2\left[y^{3/2} - y^3\right]dy = \frac{1}{3}\int_0^1\left[y^{7/2} - y^5\right]dy = \frac{1}{3}\left\{\frac{2}{9}y^{9/2} - \frac{1}{6}y^6\right\}\Big|_0^1$$

$$= \frac{1}{3}\left\{\frac{2}{9} - \frac{1}{6}\right\} = \frac{1}{3}\left\{\frac{12-9}{54}\right\} = \frac{1}{54}$$