Math 262

Spring 2008 Kollfelz

Calculus of 2D vector valued functions, of one Parameter

- Given $f: \mathbb{R} \to \mathbb{R}^2$ then $\vec{f}(t) = \vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} = f(t)\hat{\imath} + g(t)\hat{\jmath}$

Then:

$$\frac{d}{dt}\vec{r}(t) = \vec{r}\,'(t) = \dot{x}(t)\hat{\imath} + \dot{y}(t)\hat{\jmath} \equiv D_t\vec{r}(t)$$

$$\int \vec{r}(t)dt = \int [x(t)\hat{\imath} + y(t)\hat{\jmath}]dt = \left[\int x(t)dt\right]\hat{\imath} + \left[\int y(t)dt\right]dt$$

$$\int_a^b \vec{r}(t)dt = \left[\int_a^b x(t)dt\right]\hat{\imath} + \left[\int_a^b y(t)dt\right]\hat{\jmath}$$

THM 13.9 (p. 752, text)

1. $D_t[c\vec{r}(t)] = \frac{d}{dt}[c\vec{r}(t)] = c\vec{r}\,'(t)$

Proof: $\frac{d}{dt}\{c\vec{r}(t)\} = \frac{d}{dt}\{cx(t)\hat{\imath} + cy(t)\hat{\jmath}\}$

$$= \frac{d}{dt}(cx(t)\hat{\imath}) + \frac{d}{dt}(cy(t)\hat{\jmath}) = c\frac{d}{dt}x(t)\hat{\imath} + c\frac{d}{dt}y(t)\hat{\jmath}$$

$$= c\dot{x}(t)\hat{\imath} + c\dot{y}(t)\hat{\jmath} = c[\dot{x}(t)\hat{\imath} + \dot{y}(t)\hat{\jmath}] = c\vec{r}\,'(t)$$

2. $D_t[\vec{r}(t) \pm \vec{u}(t)] = D_t\vec{r}(t) \pm D_t\vec{u}(t)$

Proof: $\frac{d}{dt}[\vec{r}(t) \pm \vec{u}(t)] = \frac{d}{dt}[(r_x(t)\hat{\imath} + r_y(t)\hat{\jmath}) \pm (u_x(t)\hat{\imath} + u_y\hat{\jmath})]$

$$= \frac{d}{dt}[(r_x(t) \pm u_x(t))\hat{\imath} + (r_y(t) \pm u_y(t))\hat{\jmath}]$$

$$= \frac{d}{dt}(r_x(t) \pm u_x(t))\hat{\imath} + \frac{d}{dt}(r_y(t) \pm u_y(t))\hat{\jmath}$$

$$= (\dot{r}_x(t) \pm \dot{u}_x(t))\hat{\imath} + (\dot{r}_y(t) \pm \dot{u}_y(t))\hat{\jmath}$$

$$= [\dot{r}_x(t)\hat{\imath} + \dot{r}_y(t)\hat{\jmath}] \pm [\dot{u}_x(t)\hat{\imath} + \dot{u}_y(t)\hat{\jmath}] = D_t\vec{r}(t) + D_t\vec{u}(t)$$

③ $D_t[f(t)\vec{r}(t)] = f'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$

Proof $D_t[f(t)\vec{r}(t)] = \frac{d}{dt}\{f(t)[r_x(t)\hat{\imath} + r_y(t)\hat{\jmath}]\}$

$= \frac{d}{dt}\{f(t)r_x(t)\hat{\imath} + f(t)r_y(t)\hat{\jmath}\} = \frac{d}{dt}(f(t)r_x(t))\hat{\imath} + \frac{d}{dt}(f(t)r_y(t))\hat{\jmath}$

$= (f'(t)r_x(t) + f(t)r_x'(t))\hat{\imath} + (f'(t)r_y(t) + f(t)r_y'(t))\hat{\jmath}$

$= (f'(t)r_x(t)\hat{\imath} + f'(t)r_y(t)\hat{\jmath}) + (f(t)r_x'(t)\hat{\imath} + f(t)r_y'(t)\hat{\jmath})$

$= f'(t)[r_x(t)\hat{\imath} + r_y(t)\hat{\jmath}] + f(t)[r_x'(t)\hat{\imath} + r_y'(t)\hat{\jmath}]$

$= f'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$

④ $D_t[\vec{r}(t)\cdot\vec{u}(t)] = (D_t\vec{r}(t))\cdot\vec{u}(t) + \vec{r}(t)\cdot(D_t\vec{u}(t))$

Proof $D_t[\vec{r}(t)\cdot\vec{u}(t)] = \frac{d}{dt}\{(r_x(t)\hat{\imath} + r_y(t)\hat{\jmath})\cdot(u_x(t)\hat{\imath} + u_y(t)\hat{\jmath})\}$

$= \frac{d}{dt}\{r_x(t)u_x(t) + r_y(t)u_y(t)\} = \frac{d}{dt}(r_x(t)u_x(t) + r_y(t)u_y(t)\}$

$= r_x'(t)u_x(t) + r_x(t)u_x'(t) + r_y'(t)u_y(t) + r_y(t)u_y'(t)$

$= [r_x'(t)u_x(t) + r_y'(t)u_y(t)] + [r_x(t)u_x'(t) + r_y(t)u_y'(t)]$

$= [(r_x'(t)\hat{\imath} + r_y'(t)\hat{\jmath})\cdot(u_x(t)\hat{\imath} + u_y(t)\hat{\jmath})] + [(r_x(t)\hat{\imath} + r_y(t)\hat{\jmath})\cdot(u_x'(t)\hat{\imath} + u_y'(t)\hat{\jmath})]$

$= [D_t\vec{r}(t)\cdot\vec{u}(t)] + \vec{r}(t)\cdot[D_t\vec{u}(t)]$

(5) $D_t[\vec{r}(f(t))] = \vec{r}\,'(f(t))f'(t)$

<u>Proof</u> : $\frac{d}{dt}[\vec{r}(f(t))] = \frac{d}{dt}[r_x(f(t))\hat{\imath} + r_y(f(t))\hat{\jmath}]$

$= r_x'(f(t))f'(t)\hat{\imath} + r_y'(f(t))f'(t)\hat{\jmath}$

$= \vec{r}\,'(f(t))f'(t)$

- Example # 24, § ~~14.3~~ 13.3

$\vec{r}(t) = \langle t - \sin t, 1 - \cos t\rangle = (t - \sin t)\hat{\imath} + (1 - \cos t)\hat{\jmath}$

$\vec{r}\,'(t) = D_t(\vec{r}(t)) = \frac{d}{dt}\{(t - \sin t)\hat{\imath} + (1 - \cos t)\hat{\jmath}\}$

$= \frac{d}{dt}(t - \sin t)\hat{\imath} + \frac{d}{dt}(1 - \cos t)\hat{\jmath} = (1 - \cos t)\hat{\imath} + \sin t\,\hat{\jmath}$

$\vec{r}\,''(t) = \frac{d}{dt}\vec{r}\,'(t) = \frac{d}{dt}\{(1 - \cos t)\hat{\imath} + \sin t\,\hat{\jmath}\} = \frac{d}{dt}(1 - \cos t)\hat{\imath} + \frac{d}{dr}\sin t\,\hat{\jmath}$

$= \sin t\,\hat{\imath} + \cos t\,\hat{\jmath}$

- <u>Example # 31 §13.3</u>

$\vec{r}(t) = 3t\hat{\imath} + 4t\hat{\jmath} \qquad \vec{u}(t) = 4t\hat{\imath} + t^2\hat{\jmath}$

(a) $\vec{r}\,'(t) = \frac{d}{dt}(3t\hat{\imath} + 4t\hat{\jmath}) = 3\hat{\imath} + 4\hat{\jmath}$

(b) $D_t[\vec{r}(t)\cdot\vec{u}(t)] = \vec{r}\,'(t)\cdot\vec{u}(t) + \vec{r}(t)\cdot\vec{u}\,'(t)$

$= (3\hat{\imath} + 4\hat{\jmath})\cdot(4t\hat{\imath} + t^2\hat{\jmath}) + (3t\hat{\imath} + 4t\hat{\jmath})\cdot\frac{d}{dt}(4t\hat{\imath} + t^2\hat{\jmath})$

$= (12t + 4t^2) + (3t\hat{\imath} + 4t\hat{\jmath})\cdot(4\hat{\imath} + 2t\hat{\jmath})$

$= (12t + 4t^2) + (12t + 8t^2) = 24t + 12t^2 = 12t(2+t)$

(c) $D_t[3\vec{r}(t) - \vec{u}(t)] = \dfrac{d}{dt}\{9t\,\hat{\imath} + 12t\,\hat{\jmath} - 4t\,\hat{\imath} - t^2\hat{\jmath}\}$

$$= \frac{d}{dt}5t\,\hat{\imath} + \frac{d}{dt}(12t - t^2)\hat{\jmath} = 5\hat{\imath} + (12-2t)\hat{\jmath}$$

(d) $D_t[\|\vec{r}(t)\|] = \dfrac{d}{dt}\left[(r_x^2(t) + r_y^2(t))^{1/2}\right] = \dfrac{d}{dt}\sqrt{9t^2 + 16t^2}$

$$= \frac{d}{dt}\sqrt{25t^2} = \frac{d}{dt}5t = 5$$

• <u>Example 49 §13.3</u>

$\vec{r}''(t) = -32\hat{\jmath} \qquad \vec{r}'(0) = 600(\sqrt{3}\,\hat{\imath} + \hat{\jmath}) \qquad \vec{r}(0) = \vec{0}$

$$\vec{r}'(t) = \int \vec{r}''(t)dt = \left[\int r_x''(t)dt\right]\hat{\imath} + \left[\int r_y''(t)\,dt\right]\hat{\jmath}$$

$$= \left[\int 0\,dt\right]\hat{\imath} + \left[-32\int dt\right]\hat{\jmath} = c_1\hat{\imath} + (-32t + c_2)\hat{\jmath}$$

$$\vec{r}'(0) = 600(\sqrt{3}\,\hat{\imath} + \hat{\jmath}) = c_1\hat{\imath} + c_2\hat{\jmath}$$

$\therefore\ c_1 = 600\sqrt{3} \qquad c_2 = 600 \quad \therefore\ \vec{r}'(t) = 600\sqrt{3}\,\hat{\imath} + (-32t + 600)\hat{\jmath}$

$$\vec{r}(t) = \int \vec{r}'(t)dt = 600\sqrt{3}\int dt\,\hat{\imath} + \int(-32t + 600)dt\,\hat{\jmath}$$

$$= (600\sqrt{3}t + K_1)\hat{\imath} + (-16t^2 + 600t + K_2)\hat{\jmath}$$

$\vec{r}(0) = \vec{0} = 0\hat{\imath} + 0\hat{\jmath} = K_1\hat{\imath} + K_2\hat{\jmath} \quad \therefore\ K_1 = K_2 = 0$

$\therefore\ \vec{r}(t) = 600\sqrt{3}t\,\hat{\imath} + (-16t^2 + 600t)\hat{\jmath}$