MA262 **Calculus II: Notes for April 7**
Spring 2008 Kallfelz

- *VECTORS: 2D AND 3D*

By a *vector* in $\mathbf{R}^2$ or $\mathbf{R}^3$ (i.e., the 2-dimensionsal or 3-dimensional "Euclidean" plane or volume[1]) one is referring to a *geometric object* that has two properties: *magnitude* (or "length", characterized by the length function: $\| \ \|$) and *direction.* Conversely, mathematical objects that have only the property of *magnitude* are called *scalars.* Obviously, *numbers* are scalars.

You're about to be introduced to how Calculus can be extended into the domain of vectors. In a course like Calculus II, however, we at best scratch the surface. A full-blown treatment of Calculus using vectors in $\mathbf{R}^2$ or $\mathbf{R}^3$ is reserved for Calculus III.

To distinguish a vector from a scalar, one typically places an arrow superscript over a vector, i.e.: $\vec{u}$ is a vector, but u is a scalar.[2] *If* it's understood that one is dealing with a vector quantity $\vec{u}$, and one writes: "u" instead (without the arrow superscript), then it's understood that u refers to the vector's *magnitude*. In other words, by definition, for any vector quantity $\vec{v}$: $\|\vec{v}\| = v$.

There are many ways to represent vectors, algebraically. In 2D and 3D (i.e. vectors "living" in the Euclidean plane $\mathbf{R}^2$ or Euclidean space $\mathbf{R}^3$) one can simply represent them as *ordered pairs*, or *ordered triples*, respectively. In other words, if $\vec{u}$ and $\vec{v}$ are 2D or 3D vectors, then:

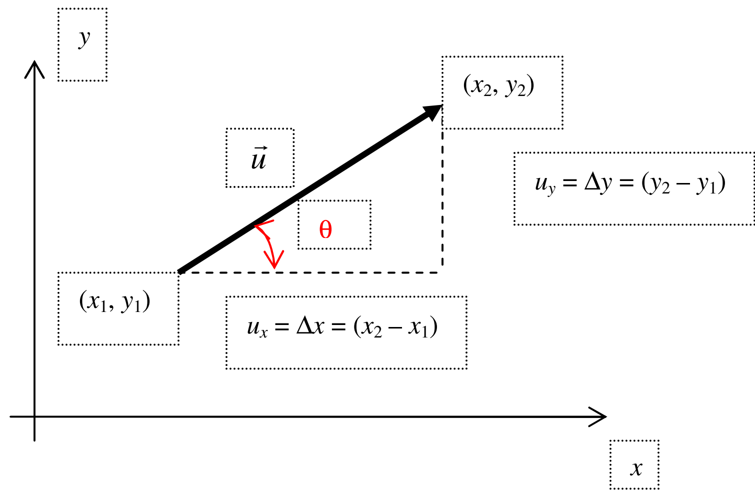
$$\vec{u} = \langle u_x, u_y \rangle \qquad \vec{v} = \langle v_x, v_y, v_z \rangle$$

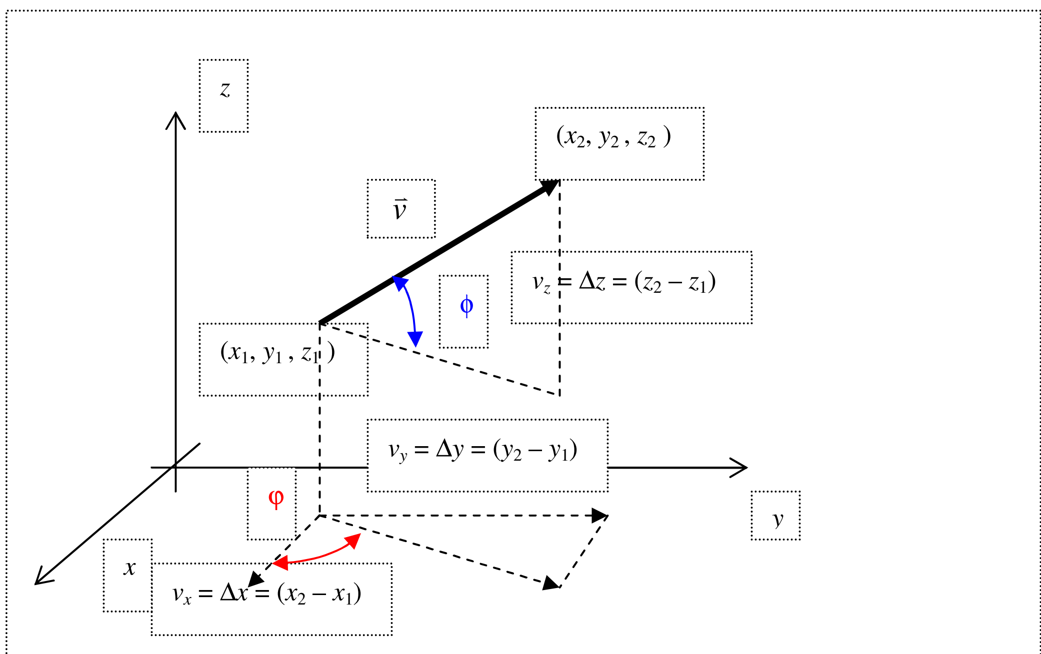
…where u_x , u_y refer to the *x*- and *y-components* of $\vec{u}$, and v_x , v_y, v_z refer to the *x-, y* and z-*components* of $\vec{v}$, respectively. This reflects the fact that any vector in a plane or a vector in space takes on a natural geometric interpretation of a *directed ray*. The

[1] The notation $\mathbf{R}^2$ or $\mathbf{R}^3$ refers to the set of all ordered real-valued ordered pairs, triples, i.e. : $\mathbf{R}^2 = \{\langle x_1, x_2 \rangle \mid x_1 \in R, x_2 \in R\}$ or $\mathbf{R}^3 = \{\langle x_1, x_2, x_3 \rangle \mid x_1 \in R, x_2 \in R, x_3 \in R\}$. One calls such spaces "Euclidean" since as we have seen, a set of real-valued ordered pairs takes on a natural geometric interpretation of representing the Cartesian plane. The set of all real-valued triples assumes the same natural interpretation representing the 3-D Cartesian 'box'. These coordinate systems naturally correspond to the "commonsense" Euclidean intuition that space and geometry is ultimately best characterized by straight lines, flat surfaces, simple volumes (boxes, etc). Of course Einstein, Riemann, and Lovachesky showed that there are cases in which such conceptions are inadequate: I.e. geometries (including those representing physical space) can be "curved," or characterized by varying measures of "local straightness" and "perpendicularity"—phrased more technically, by a varying metric, as opposed to a constant metric in the Euclidean case. The terms are "straightness" and "perpendicularity" are deliberately placed in quotes to indicate that there's no *absolute* way to characterize straightness and perpendicularity in the general case in which space is "curved," or characterized by a varying metric. This notion is what lies at the root of the theory of General Relativity, for instance.

[2] Vectors are also denoted in **boldface** notation. To avoid excessive proliferation of different notations representing the same thing, I'll stick with the arrow superscript notation.

components denote the *change* in the x, y direction (in the 2D case) as well as the change of the x, y, z direction (in the 3D case), as illustrated in the figures below:





- **Note 1:** The x, y, z coordinate system has positive directions for x, y, z that is based on the *right-hand convention* (how, for instance, most screws are threaded). Take your right hand, and point your index finger in the direction of $+x$ (positive x-direction) and point your middle finger in the $+y$ direction. Your thumb will point in the $+z$ direction. Note further that this *right-hand-rule* (RHR) is how we specify the direction of the cross-product in $\mathbf{R}^3$.

So, the component form is just one way to represent a two (or three) dimensional vector. Other ways include the *column-matrix* approach:

$$\vec{u} = \begin{pmatrix} u_x \\ u_y \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

…for the 2D and 3D vectors $\vec{u}, \vec{v}$ as described above. In addition, note the *inscribed angles* depicted in the above illustrations (in red for the 2D case and red & blue in the 3D case). Recall the *Polar Coordinate*[3] system previously discussed. One can likewise represent vectors $\vec{u}, \vec{v}$ in (circular and spherical) polar form:

$$\vec{u} = \langle u, \theta \rangle \qquad \vec{v} = \langle v, \phi, \varphi \rangle$$

where, as mentioned earlier, u and v are the respective *magnitudes* ("lengths") of vectors $\vec{u}, \vec{v}$ respectively: $\|\vec{u}\| = u, \|\vec{v}\| = v$. The angle θ in the above illustration of course is immediately recognized as the *polar angle*. Whereas the angles ϕ, φ are immediately recognized as the *altitude* and *azimuth* (think of the case of the measure of *latitude* and longitude).

Of course, by virtue of the Pythagorean Theorem, the length or magnitude of the vectors $\vec{u}, \vec{v}$ are computed as follows:

$$\|\vec{u}\| = u = \sqrt{u_x^2 + u_y^2} = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\|\vec{v}\| = v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

This introduces the special case of a *unit vector*, or vectors with magnitude = 1. Any unit vector is denoted by the tile (" ^ ") superscript, i.e. for every $\hat{u}$, then by definition: $\|\hat{u}\| = 1$. *Every* non-zero vector $\vec{u}$ can be depicted in "standard form" in the following fashion: $\vec{u} = u\hat{u}$.

[3] The difference however, between the polar coordinate system and the polar representation of vectors, is that recall in the polar system r (the first entry in the ordered pair (r, θ) can take on *any* real value (both negative or positive). Whereas, in the polar representation of vectors, by definition since the first entry in the ordered pair/triple represents the vector's *length* or *magnitude*, then it must be nonnegative.

The associated unit vector $\hat{u}$ is related to $\vec{u}$ by simply dividing $\vec{u}$ by its magnitude, i.e.: $\hat{u} = \frac{\vec{u}}{\|\vec{u}\|} = \left(\frac{1}{u}\right)\vec{u} = \left(u^{-1}\right)\vec{u}$. For instance, consider the 2D vector $\vec{u} = \langle -2,5 \rangle$. Then its associated unit vector is: $\hat{u} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{\langle -2,5 \rangle}{\sqrt{(-2)^2 + 5^2}} = \frac{1}{\sqrt{29}}\langle -2,5 \rangle = \left\langle -\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle$. This can be easily checked: $\|\hat{u}\| = \sqrt{\hat{u}_x^2 + \hat{u}_y^2} = \sqrt{\left(-\frac{2}{\sqrt{29}}\right)^2 + \left(\frac{5}{\sqrt{29}}\right)^2} = \sqrt{\frac{4+25}{29}} = 1$

Unit vectors say something about a vector's *direction*. They act as "pointers." So the decomposition $\vec{u} = u\hat{u}$ is a natural way of characterizing a vector via its two basic properties (magnitude and direction), since the above product represents a product of length and direction.

It is easily seen that the $\vec{u} = u\hat{u}$ factoring always holds for *non-zero* vectors, (i.e. vectors of non-zero length). Consider for example some 3D nonzero vector:

$$\vec{v} = \langle v_x, v_y, v_z \rangle = \left\langle \|v\|\frac{v_x}{\|v\|}, \|v\|\frac{v_y}{\|v\|}, \|v\|\frac{v_z}{\|v\|} \right\rangle = \|v\|\left\langle \frac{v_x}{\|v\|}, \frac{v_y}{\|v\|}, \frac{v_z}{\|v\|} \right\rangle = v\left\langle \frac{v_x}{v}, \frac{v_y}{v}, \frac{v_z}{v} \right\rangle$$

(Note how the 'trick of 1' was used in the first step). Now, the three components form a unit vector, as is easily shown:

$$\sqrt{\left(\frac{v_x}{v}\right)^2 + \left(\frac{v_y}{v}\right)^2 + \left(\frac{v_z}{v}\right)^2} = \sqrt{\frac{v_x^2 + v_y^2 + v_z^2}{v^2}} = \sqrt{\frac{\|\vec{v}\|^2}{v^2}} = \sqrt{\frac{v^2}{v^2}} = 1$$

Therefore: $\vec{v} = \langle v_x, v_y, v_z \rangle = v\left\langle \frac{v_x}{v}, \frac{v_y}{v}, \frac{v_z}{v} \right\rangle = v\hat{v}$

- *ALGEBRAIC OPERATIONS ON VECTORS*

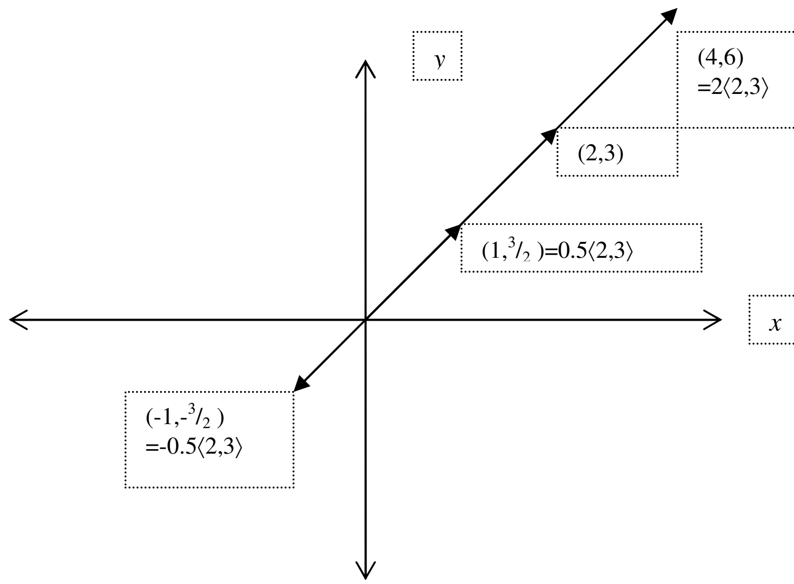
  The above decomposition: $\vec{v} = \langle v_x, v_y, v_z \rangle = v\left\langle \frac{v_x}{v}, \frac{v_y}{v}, \frac{v_z}{v} \right\rangle = v\hat{v}$ is an example of *scalar multiplication* (since length is a magnitude, i.e. a non-negative scalar). Scalar multiplication is defined formally as:

  For any real number *c* (or scalar) and any 2D or 3D vectors $\vec{u}, \vec{v}$:

  $c\vec{u} = c\langle u_x, u_y \rangle = \langle cu_x, cu_y \rangle$

  $c\vec{v} = c\langle v_x, v_y, v_z \rangle = \langle cv_x, cv_y, cv_z \rangle$

So one recognizes the rule of scalar multiplication as simply distributing across its components by the scale factor *c*. Recall the previous discussion of the natural decomposition: $\vec{u} = u\hat{u}$. Then: $c\vec{u} = c(u\hat{u}) = (cu)\hat{u}$. I.e., interpreted geometrically, *c* affects the *length* or *magnitude* of the vector (whenever $c > 0$). If $c < 0$, then in addition to altering the vector's length, *c* rotates the vector by 180^o. Nevertheless, the vector still lies on the same line of action. Consider the 2D vector : $\vec{u} = \langle 2,3 \rangle$. The illustration below indicates the action of *c* on $\vec{u} = \langle 2,3 \rangle$ for the cases: $c = 0.5$, $c = 2$, $c = -0.5$:



In addition, two (or more) vectors can be added together. Addition is defined via *component addition*:

For any 2D or 3D pairs of vectors $\vec{u}, \vec{v}$:

$$\vec{u} + \vec{v} = \langle u_x, u_y \rangle + \langle v_x, v_y \rangle = \langle u_x + v_x, u_y + v_y \rangle$$
$$\vec{u} + \vec{v} = \langle u_x, u_y, u_z \rangle + \langle v_x, v_y, v_z \rangle = \langle u_x + v_x, u_y + v_y, u_z + v_z \rangle$$

Based on the above definition of scalar multiplication, vector *subtraction* is easily defined:

For any 2D or 3D pairs of vectors $\vec{u}, \vec{v}$:

$$\vec{u}-\vec{v}=\vec{u}+(-1)\vec{v}=\left\langle u_x,u_y\right\rangle+\left\langle -v_x,-v_y\right\rangle=\left\langle u_x-v_x,u_y-v_y\right\rangle$$
$$\vec{u}-\vec{v}=\vec{u}+(-1)\vec{v}=\left\langle u_x,u_y,u_z\right\rangle+\left\langle -v_x,-v_y,-v_z\right\rangle=\left\langle u_x-v_x,u_y-v_y,u_z-v_z\right\rangle$$

I.e., as simply componentwise subtracation. As discussed in class, and in Fig. 13.7, text (p. 731), vector addition/subtraction is represented by Paralellogram Rule: The long diagonal of the plane spanned by $\vec{u},\vec{v}$ (whether in space or in the plane, i.e. whether these vector-pairs are 3D or 2D) is the sum $\vec{u}+\vec{v}$, whereas the short diagonal is the difference: $\vec{u}-\vec{v}$.

Scalar multiplication, vector addition and subtraction form a naturally closed set of properties characterizing what's known as a *vector space.* Its basic characteristics are summarized in Thm 13.1, p. 731 (commutativity and associativity of addition, additive identity and cancellation, two kinds of distributivity, and aobsorbtion by 1 and 0 (i.e. property 8.))

- *THE HAMILTIONIAN NOTATION*

In addition to the ordered pair/triple, column-vector, and polar notation depicted above, the properties of scalar multiplication and addition can be exploited to produced yet another way of conveniently characterizing vectors. Observe in the case of the 3D vector: $\vec{v}=\left\langle v_x,v_y,v_z\right\rangle=\left\langle v_x,0,0\right\rangle+\left\langle 0,v_y,v_z\right\rangle=\left\langle v_x,0,0\right\rangle+\left\langle 0,v_y,0\right\rangle+\left\langle 0,0,v_z\right\rangle$

This natural decomposition is of course always assured by the definition of vector addition. Furthermore, note that by scalar mulitiplication:

$$\vec{v}=\left\langle v_x,0,0\right\rangle+\left\langle 0,v_y,0\right\rangle+\left\langle 0,0,v_z\right\rangle=v_x\left\langle 1,0,0\right\rangle+v_y\left\langle 0,1,0\right\rangle+v_z\left\langle 0,0,1\right\rangle$$

Now certainly the vectors: $\langle 1,0,0\rangle,\langle 0,1,0\rangle,\langle 0,0,1\rangle$ are unit vectors, pointing in the positive *x*, *y*, *z* directions, respectively. Denote them by: $\hat{i}=\langle 1,0,0\rangle,\hat{j}=\langle 0,1,0\rangle,\hat{k}=\langle 0,0,1\rangle$. Hence we may write: $\vec{v}=\left\langle v_x,v_y,v_z\right\rangle=v_x\hat{i}+v_y\hat{j}+v_z\hat{k}$

And for any 2D vector of course: $\vec{u}=\left\langle u_x,u_y\right\rangle=u_x\hat{i}+u_y\hat{j}$

Recall the natural decomposition discussed earlier: $\vec{u}=u\hat{u}$. A natural characterization of the associated unit vector $\hat{u}$ is obtained in the Hamiltonian notation which brings out the information concerning a vector's polar angle θ (in the 2D case). Observe:

$$\vec{u}=u_x\hat{i}+u_y\hat{j}=u\left(\frac{u_x}{u}\hat{i}+\frac{u_y}{u}\hat{j}\right)$$

However, based on the above illustration in the 2D case, one sees immediately that in terms of the vector's polar angle θ: $\frac{u_x}{u} = \cos\theta, \frac{u_y}{u} = \sin\theta$. Hence:

$$\vec{u} = u_x\hat{i} + u_y\hat{j} = u\left(\frac{u_x}{u}\hat{i} + \frac{u_y}{u}\hat{j}\right) = u(\cos\theta\hat{i} + \sin\theta\hat{j}) = u\hat{u}$$

Hence any non-zero 2D vector $\vec{v}$ has a unit vector which can naturally be expressed by:

$$\hat{v} = \cos\theta\hat{i} + \sin\theta\hat{j} = \langle\cos\theta, \sin\theta\rangle$$

It's obvious that it's a unit vector, since: $\|\hat{v}\| = \sqrt{(\cos\theta)^2 + (\sin\theta)^2} = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

The same reasoning holds for the case of a 3D vector:

$\vec{u} = u_x\hat{i} + u_y\hat{j} + u_z\hat{k} = u\left(\frac{u_x}{u}\hat{i} + \frac{u_y}{u}\hat{j} + \frac{u_z}{u}\hat{k}\right)$. Now: $\frac{u_x}{u} = \cos\alpha, \frac{u_y}{u} = \cos\beta, \frac{u_z}{u} = \cos\gamma$,

where α, β, γ are the *direction cosines* of the vector $\vec{u}$, i.e. when inscribed in a rectangular solid (box) in which $\vec{u}$ is drawn from one opposite corner of the box to the other (in such a way that it diagonally bisects the box), three angles are naturally defined whose cosines are represented by the ratios of the *x*, *y*, *z* components of the vector with respect to the vector's length. For an illustration, see Fig. 14.11, p. 791, text. Hence:

$$\vec{u} = u_x\hat{i} + u_y\hat{j} + u_z\hat{k} = u\left(\frac{u_x}{u}\hat{i} + \frac{u_y}{u}\hat{j} + \frac{u_z}{u}\hat{k}\right) = u(\cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}) = u\hat{u}$$

Note an interesting feature of these direction cosines (which are *not* to be confused with the altitude and azimuth angles ϕ, φ discussed above. Since $\hat{u}$ is a unit vector, then:

$$\|\hat{u}\| = \sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} = 1 \Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

…which means that *only two of the three direction cosines can be independently chosen*. Suppose for instance one chooses independent values for α, β. Then γ is constrained by the above, i.e. it has "no choice" but to take on the value:

$$\cos^2\gamma = 1 - \cos^2\alpha - \cos^2\beta \Rightarrow \gamma = \arccos(1 - \cos^2\alpha - \cos^2\beta)$$

Hence, even though direction cosines are *not* the same as altitude and azimuth angles, the essentially geometric fact is the same in both cases. Just as in the 2D plane, *one* angle is sufficient to specify the direction for a vector (i.e. the vector's polar angles), so in 3D space, *only two* angles are necessary to fix the direction of a vector (whether they're the altitude and azimuth angles, or two out of three of the vector's direction cosines, etc.)

- *THE DOT PRODUCT*

Given any pair of vectors $\vec{u},\vec{v}$ (whether 2D or 3D), their *dot product* is defined by:

$$\vec{u}\cdot\vec{v}=\langle u_x,u_y\rangle\cdot\langle v_x,v_y\rangle=(u_x\hat{i}+u_y\hat{j})\cdot(v_x\hat{i}+v_y\hat{j})=u_xv_x+u_yv_y$$
$$\vec{u}\cdot\vec{v}=\langle u_x,u_y,u\rangle\cdot\langle v_x,v_y,v_z\rangle=(u_x\hat{i}+u_y\hat{j}+u_z\hat{k})\cdot(v_x\hat{i}+v_y\hat{j}+v_z\hat{k})=u_xv_x+u_yv_y+u_zv_z$$

- **Property 1** The dot product is clearly *commutative*, since (in the 2D case)

  Proof
  $\vec{u}\cdot\vec{v}=u_xv_x+u_yv_y=v_xu_x+v_yu_y=\vec{v}\cdot\vec{u}$

  (obviously the same demonstration holds in the 3D case)

- **Property 2:** For any vector (2D or 3D): $\vec{u}\cdot\vec{u}=\|\vec{u}\|^2=u^2$
  Proof
  Consider for example the case of a 3D vector:

  $$\vec{u}\cdot\vec{u}=u_x^2+u_y^2+u_z^2=\left(\sqrt{u_x^2+u_y^2+u_z^2}\right)^2=(\|\vec{u}\|)^2=\|\vec{u}\|^2=u^2$$

- **Property 3:** (Geometric interpretation of the dot product) Given $\vec{u},\vec{v}$ (whether 2D or 3D), and angle θ they subtend[4]: $\vec{u}\cdot\vec{v}=\|\vec{u}\|\|\vec{v}\|\cos\theta=uv\cos\theta$

  The text presents a derivation of this result, using the law of cosines (p. 739)

  Note how the above expression guarantees that $\hat{i},\hat{j},\hat{k}$ form a mutually orthogonal set. For by definition of the dot product (in the 2D) case:

  $$\vec{u}\cdot\vec{v}=(u_x\hat{i}+u_y\hat{j})\cdot(v_x\hat{i}+v_y\hat{j})=u_xv_x(\hat{i}\cdot\hat{i})+2u_xv_y(\hat{i}\cdot\hat{j})+u_yv_y(\hat{j}\cdot\hat{j})=u_xv_x+u_yv_y$$

  ..which implies : $\hat{i}\cdot\hat{j}=0=\cos\theta\Rightarrow\theta=\arccos 0=\frac{\pi}{2}$

  (Which substantiates the definition of $\hat{i},\hat{j},\hat{k}$ as parallel to three mutually orthogonal axes)

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[4] In $\mathbf{R^3}$**,** i.e. in the 3D case, the two vectors obviously form a plane in space, as opposed to a plane embedded in the Cartesian 2D plane.

- **Property 4** (Thm 13.5.2) : Distributivity: Given $\vec{u}, \vec{v}, \vec{w}$ (whether 2D or 3D):
$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u} = (\vec{v} + \vec{w}) \cdot \vec{u}$$

(Note how the previous property of Commutativity is tacked on). This is easy to established componentwise using 2D vectors (the procedure using 3D vectors is likewise the same)

Proof

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \langle u_x, u_y \rangle \cdot (\langle v_x, v_y \rangle + \langle w_x, w_y \rangle) = \langle u_x, u_y \rangle \cdot \langle (v_x + w_x), (v_y + w_y) \rangle$$
$$= u_x(v_x + w_x) + u_y(v_y + w_y) = u_x v_x + u_x w_x + u_y v_y + u_y w_y = u_x v_x + u_y v_y + u_x w_x + u_y w_y$$
$$= (u_x v_v + u_y v_y) + (u_x w_x + u_y w_y) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

(Certainly one can set up this using the Hamilton notation as well)

- **Property 5** (Thm 13.5.3)Given $c, \vec{v}, \vec{w}$ (c is a scalar): $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$

Proof:
This naturally follows from commutativity and associativity : Using the 2D case:

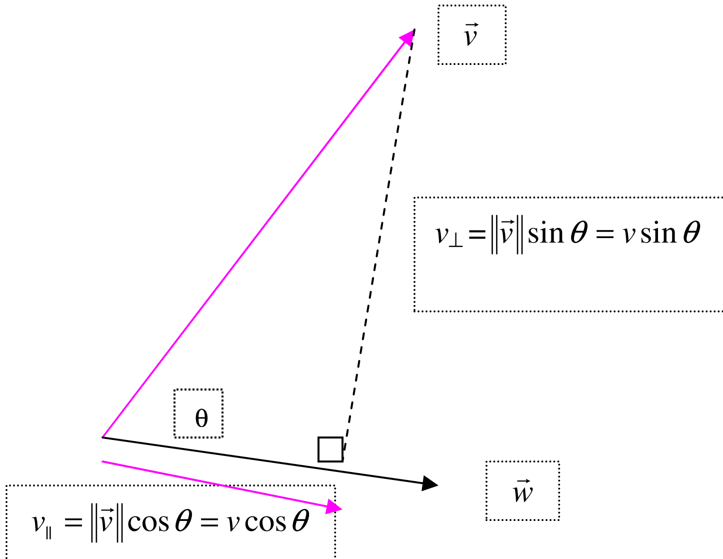
$$c(\vec{u} \cdot \vec{v}) = c(u_x v_x + u_y v_y) = cu_x v_x + cu_y v_y = (cu_x)v_x + (cu_y)v_y = (c\vec{u}) \cdot \vec{v}$$

Moreover,

$$c(\vec{u} \cdot \vec{v}) = cu_x v_x + cu_y v_y = u_x cv_x + u_y cv_y = u_x(cv_x) + u_y(cv_y)$$
$$= \vec{u} \cdot (c\vec{v})$$

- **Property 6**: Given any two vector $\vec{v}, \vec{w}$ (whether 2D or 3D), the *projection of* $\vec{v}$ in the direction of $\vec{w}$, denoted: $\mathrm{Pr}\,oj_{\vec{w}}\vec{v}$ is: $\mathrm{Pr}\,oj_{\vec{w}}\vec{v} = \left(\dfrac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} \right) \vec{w}$

Proof : As suggested by the figure below, certainly the *magnitude* of the component of $\vec{v}$ in the direction of $\vec{w}$ is, i.e: $v_{\parallel} = \|\vec{v}\| \cos\theta = v\cos\theta$, where θ is the angle made between $\vec{v}$ & $\vec{w}$, which can be easily derived by elementary trigonometry



But according to the definition of dot product, $v_{\parallel} = v\cos\theta = \vec{v}\cdot\hat{w} = \|\vec{v}\|\|\hat{w}\|\cos\theta = \|\vec{v}\|\cos\theta$ since $\hat{w}$ is $\vec{w}$'s associated unit vector and hence has magnitude = 1.

Hence the *vector* that points in $\vec{w}$'s direction with the above magnitude is: $v_{\parallel}\hat{w} = (\vec{v}\cdot\hat{w})\hat{w}$

So: $\Pr oj_{\vec{w}}\vec{v} = (\vec{v}\cdot\hat{w})\hat{w} = \left(\vec{v}\cdot\dfrac{\vec{w}}{\|\vec{w}\|}\right)\dfrac{\vec{w}}{\|\vec{w}\|} = \left(\dfrac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|^2}\right)\vec{w}$ (using the definition of unit vector, i.e.: $\hat{w} = \dfrac{\vec{w}}{\|\vec{w}\|}$

Moreover, since the vector sum of the component parallel and perpendicular to $\vec{w}$ will of course produce $\vec{v}$, hence: $\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp} \Rightarrow \vec{v}_{\perp} = \vec{v} - \vec{v}_{\parallel} = \vec{v} - \left(\dfrac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|^2}\right)\vec{w}$

- *THE CROSS PRODUCT*

While the dot product produces a *scalar*, or a number, the *cross product* between two vectors produces *another* vector. This third vector, denoted $\vec{v}\times\vec{w}$ (if the product is between $\vec{v}$ *and then* $\vec{w}$) points in a direction perpendicular to the plane defined by $\vec{v}$ and $\vec{w}$. The direction of $\vec{v}\times\vec{w}$ is determined by the RHR (recall Note 1 in page 2 above): point your index finger of your right hand in the direction of $\vec{v}$ and your middle finger in the direction of $\vec{w}$. Then you thumb will point in the direction of $\vec{v}\times\vec{w}$.

- Note 2: Because of the directionality of the cross product, it is *not* commutative! In other words, $\vec{v}\times\vec{w}$ points in a direction opposite to that of $\vec{w}\times\vec{v}$, i.e.: $\vec{v}\times\vec{w}=-\vec{w}\times\vec{v}$.
- Note 3: The cross product defines a natural family of three vectors all mutually perpendicular to one another. For that reason, the cross product is *only* defined for two vectors in $\mathbf{R}^3$**,** i.e. two 3D vectors. If these two vectors happen both to be in the *x*, *y* plane, and hence can just as well be represented by two 2D vectors, then their cross-product points either in the +*z* or –*z* direction (or in other words points in a direction perpendicular to the *x*, *y* plane). So we still need three dimensions to describe it properly, even in this special case.

Given any pair of vectors $\vec{u},\vec{v}$ (3D), their *cross- product* is formed by first taking all possible products:

$$\vec{u}\times\vec{v}=\langle u_x,u_y,u\rangle\times\langle v_x,v_y,v_z\rangle=\left(u_x\hat{i}+u_y\hat{j}+u_z\hat{k}\right)\times\left(v_x\hat{i}+v_y\hat{j}+v_z\hat{k}\right)$$
$$=u_xv_x\left(\hat{i}\times\hat{i}\right)+u_xv_y\left(\hat{i}\times\hat{j}\right)+u_xv_z\left(\hat{i}\times\hat{k}\right)+u_yv_x\left(\hat{j}\times\hat{i}\right)+u_yv_y\left(\hat{j}\times\hat{j}\right)+u_yv_z\left(\hat{j}\times\hat{k}\right)+u_zv_x\left(\hat{k}\times\hat{i}\right)$$
$$+u_zv_y\left(\hat{k}\times\hat{j}\right)+u_zv_z\left(\hat{k}\times\hat{k}\right)$$

Now, as mentioned, $\hat{i},\hat{j},\hat{k}$ form a right-handed system. The cross-product is defined by the RHR. Therefore: $\hat{i}\times\hat{j}=\hat{k}=-\hat{j}\times\hat{i},\ \hat{j}\times\hat{k}=\hat{i}=-\hat{k}\times\hat{j},\hat{k}\times\hat{i}=\hat{j}=-\hat{i}\times\hat{k}$. Furthermore $\hat{i}\times\hat{i}=\vec{0}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}$, since forming the cross-product of a vector with itself means there is no plane spanned by a vector and itself, so no vector can result that's perpendicular from this non-existent plane.

Hence:

$$\vec{u}\times\vec{v}=u_xv_x\left(\hat{i}\times\hat{i}\right)+u_xv_y\left(\hat{i}\times\hat{j}\right)+u_xv_z\left(\hat{i}\times\hat{k}\right)+u_yv_x\left(\hat{j}\times\hat{i}\right)+u_yv_y\left(\hat{j}\times\hat{j}\right)+u_yv_z\left(\hat{j}\times\hat{k}\right)+u_zv_x\left(\hat{k}\times\hat{i}\right)$$
$$+u_zv_y\left(\hat{k}\times\hat{j}\right)+u_zv_z\left(\hat{k}\times\hat{k}\right)$$
$$=u_xv_y\hat{k}-u_xv_z\hat{j}-u_yv_x\hat{k}+u_yv_z\hat{i}+u_zv_x\hat{j}-u_zv_y\hat{i}=\left(u_yv_z-u_zv_y\right)\hat{i}+\left(u_zv_x-u_xv_z\right)\hat{j}+\left(u_xv_y-u_yv_x\right)\hat{k}$$
$$=\left(u_xv_y-u_yv_x\right)\hat{k}-\left(u_xv_z-u_zv_x\right)\hat{j}+\left(u_yv_z-u_zv_y\right)\hat{i}$$
$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ u_x & u_y & u_z\\ v_x & v_y & v_z\end{vmatrix}$$

…where the above represents the determinant of a 3×3 matrix (for further details, see p. 796, text) Based on the above derivation, the properties in Thm 14.1 (p. 797, text) can all be easily obtained:

- **Property 1:** $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.
  Proof

  This was already *geometrically* established in the above remarks concerning the RHR. To show it *algebraically*, note that:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} \qquad \vec{v} \times \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ u_x & u_y & u_z \end{vmatrix}$$

  Swapping any two rows/collums of a matrix will change the sign of its determinant, hence: $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

- **Property 2:** $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$ (distributivity)

Proof

$$\vec{u} \times (\vec{v} + \vec{w}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_x & u_y & u_z \\ v_x + w_x & v_y + w_y & v_z + w_z \end{vmatrix} = u_y (v_z + w_z)\hat{i} - (v_y + w_y)u_z \hat{i} - u_x (v_z + w_z)\hat{j} + (v_x + w_x)u_y \hat{j}$$

$$+ u_x (v_y + w_y)\hat{k} - (v_x + w_x)u_y \hat{k} = \left[(u_y v_z - v_y u_z)\hat{i} - (u_x v_z - v_x u_z)\hat{j} + (u_x v_y - v_x u_y)\hat{k}\right]$$
$$+ \left[(u_y w_z - w_y u_z)\hat{i} - (u_x w_x - w_x u_z)\hat{j} + (u_x w_y - w_x u_y)\hat{k}\right]$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_x & u_y & u_z \\ w_x & w_y & w_z \end{vmatrix} = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$$

- **Property 3:** $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$ (for any scalar c)

Proof : Multiplying the determinant of a matrix by any constant c is equivalent to taking the determinant of the matrix in which c is multiplied across by any of its row/columns. I.e.:

$$c\vec{u} \times \vec{v} = c\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ cu_x & cu_y & cu_z \\ v_x & v_y & v_z \end{vmatrix} = (c\vec{u}) \times \vec{v}$$

$$c\vec{u}\times\vec{v}=c\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ u_x & u_y & u_z\\ v_x & v_y & v_z\end{vmatrix}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ u_x & u_y & u_z\\ cv_x & cv_y & cv_z\end{vmatrix}=\vec{u}\times(c\vec{v})$$

Therefore: $c(\vec{u}\times\vec{v})=(c\vec{u})\times\vec{v}=\vec{u}\times(c\vec{v})$

- **Property 4**: $(\vec{0}\times\vec{v})=\vec{0}=\vec{v}\times\vec{0}$

Proof: Any matrix with all zeros for any of its rows or columns will produce a zero determninant

- **Property 5**: $(\vec{v}\times\vec{v})=\vec{0}$

Proof : Though this was already *geoemetrically* established via the RHR above, *algebraically* this property is obtained because any determinant of a matrix with two equal rows/columns will = 0

- **Property 6**: $\vec{u}\cdot(\vec{v}\times\vec{w})=(\vec{u}\times\vec{v})\cdot\vec{w}$

Proof: $\vec{u}\cdot(\vec{v}\times\vec{w})=\left(u_x\hat{i}+u_y\hat{j}+u_z\hat{k}\right)\cdot\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ v_x & v_y & v_z\\ w_x & w_y & w_z\end{vmatrix}=\begin{vmatrix}u_x & u_y & u_z\\ v_x & v_y & v_z\\ w_x & w_y & w_z\end{vmatrix}$

$$(\vec{u}\times\vec{v})\cdot\vec{w}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ u_x & u_y & u_z\\ v_x & v_y & v_z\end{vmatrix}\cdot\left(w_x\hat{i}+w_y\hat{j}+w_z\hat{k}\right)=\begin{vmatrix}w_x & w_y & w_z\\ u_x & u_y & u_z\\ v_x & v_y & v_z\end{vmatrix}$$

Now, the second matrix is an *even permutation* of the first matrix. That is to say, by an even number of exchanges of rows (or columns) one can derive one from the other. Notice the second matrix is obtained by first swapping row 1 with row 2, and then swapping row 3 with row 1. Determinants of matrices which are even permutations of one another are the same, hence:

$$\vec{u}\cdot(\vec{v}\times\vec{w})=\begin{vmatrix}u_x & u_y & u_z\\ v_x & v_y & v_z\\ w_x & w_y & w_z\end{vmatrix}=\begin{vmatrix}w_x & w_y & w_z\\ u_x & u_y & u_z\\ v_x & v_y & v_z\end{vmatrix}=(\vec{u}\times\vec{v})\cdot\vec{w}$$

As discussed in Thm 14.3 (p. 800), the above is a *triple scalar product*, representing the volume of a parallepiped spanned by the three vectors $\vec{u},\vec{v},\vec{w}$. As shown also (Thm 14.2) $\|\vec{u}\times\vec{v}\|=\|\vec{u}\|\|\vec{v}\||\sin\theta|$, where θ is the angle subtended by the two vectors

- <u>Example: # 55, §13.1</u>

$\|\vec{u}\| = 1, \alpha = \frac{\pi}{4}, \|\vec{u} + \vec{v}\| = \sqrt{2}, \beta = \frac{\pi}{2}$

$\therefore \vec{u} = u\hat{u} = u(\cos\alpha\hat{i} + \sin\alpha\hat{j}) = (\cos\frac{\pi}{4}\hat{i} + \sin\frac{\pi}{4}\hat{j}) = \frac{\sqrt{2}}{2}(\hat{i} + \hat{j}) = u_x\hat{i} + u_y\hat{j}$

$\therefore \vec{u} + \vec{v} = (u + v)(\hat{u} + \hat{v}) = (u + v)(\cos\beta\hat{i} + \sin\beta\hat{j}) = \sqrt{2}(\cos\frac{\pi}{2}\hat{i} + \sin\frac{\pi}{2}\hat{j}) = \sqrt{2}(0\hat{i} + \hat{j}) =$
$(u_x + v_x)\hat{i} + (u_y + v_y)\hat{j}$

Hence equating the x and y components:

$u_x = \frac{\sqrt{2}}{2}, u_x + v_x = 0 \Rightarrow v_x = -u_x = -\frac{\sqrt{2}}{2}$

$u_y = \frac{\sqrt{2}}{2}, u_y + v_y = \sqrt{2} \Rightarrow v_y = \sqrt{2} - u_y = \frac{\sqrt{2}}{2}$

$\therefore \vec{v} = v_x\hat{i} + v_y\hat{j} = \frac{\sqrt{2}}{2}(-\hat{i} + \hat{j})$

- <u>Example: # 29, §13.2</u>

$\vec{u} = \langle 2,-3 \rangle = 2\hat{i} - 3\hat{j} \qquad \vec{v} = \langle 3,2 \rangle = 3\hat{i} + 2\hat{j}$

a.) $\mathrm{Proj}_{\vec{v}}\vec{u} = \vec{u}_{\|} = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\right)\vec{v} = \left(\frac{2 \cdot 3 - 3 \cdot 2}{3^2 + 2^2}\right)(3\hat{i} + 2\hat{j}) = \vec{0}$

b.) $\vec{u}_{\perp} = \vec{u} - \vec{u}_{\|} = \vec{u} = 2\hat{i} - 3\hat{j}$

- <u>Example: # 38, §13.2</u>

$\|\vec{u} - \vec{v}\|^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = (\vec{u} \cdot \vec{u}) - (\vec{v} \cdot \vec{u}) - (\vec{u} \cdot \vec{v}) + (\vec{v} \cdot \vec{v})$

By commutativity of dot product and **Property 2:**

$\|\vec{u} - \vec{v}\|^2 = (\vec{u} \cdot \vec{u}) - 2(\vec{u} \cdot \vec{v}) + (\vec{v} \cdot \vec{v}) = \|\vec{u}\|^2 - 2(\vec{u} \cdot \vec{v}) + \|\vec{v}\|^2$

- <u>Example: # 39, §13.2</u>

$|\vec{u} \cdot \vec{v}| = \left|\|\vec{u}\|\|\vec{v}\|\cos\theta\right| = \|\vec{u}\|\|\vec{v}\||\cos\theta|$**.** However, since: $0 \le |\cos\theta| \le 1$

Hence $\left|\vec{u}\cdot\vec{v}\right| = \left\|\vec{u}\right\|\left\|\vec{v}\right\|\left|\cos\theta\right| \le \left\|\vec{u}\right\|\left\|\vec{v}\right\|$

- Example: # 40, §13.2

$$\left\|\vec{u}+\vec{v}\right\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = (\vec{u}\cdot\vec{u}) + 2(\vec{u}\cdot\vec{v}) + (\vec{v}\cdot\vec{v}) = \left\|\vec{u}\right\|^2 + 2(\vec{u}\cdot\vec{v}) + \left\|\vec{v}\right\|^2$$

However, as shown in #39 above: $\vec{u}\cdot\vec{v} \le \left|\vec{u}\cdot\vec{v}\right| \le \left\|\vec{u}\right\|\left\|\vec{v}\right\|$

Hence:

$$\left\|\vec{u}+\vec{v}\right\|^2 = \left\|\vec{u}\right\|^2 + 2(\vec{u}\cdot\vec{v}) + \left\|\vec{v}\right\|^2 \le \left\|\vec{u}\right\|^2 + 2\left\|\vec{u}\right\|\left\|\vec{v}\right\| + \left\|\vec{v}\right\|^2 = \left(\left\|\vec{u}\right\| + \left\|\vec{v}\right\|\right)^2$$

$$\therefore \left\|\vec{u}+\vec{v}\right\| \le \left\|\vec{u}\right\| + \left\|\vec{v}\right\|$$

- Example: # 73, §14.1

$\vec{u} = \langle 2,-3,1\rangle = 2\hat{i} - 3\hat{j} + \hat{k}$ $\qquad$ $\vec{v} = \langle -1,-1,-1\rangle = -\hat{i} - \hat{j} - \hat{k}$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\left\|\vec{u}\right\|\left\|\vec{v}\right\|} = \frac{-2+3-1}{\sqrt{2^2+3^2+1^2}\sqrt{1^2+1^2+1^2}} = \frac{0}{\sqrt{42}} = 0 \Rightarrow \theta = \arccos 0 = \tfrac{\pi}{2},$$

perpendicular

- Example: # 74, §14.1

$\vec{u} = \langle \cos\theta, \sin\theta, -1\rangle = \cos\theta\hat{i} + \sin\theta\hat{j} - \hat{k}$ $\qquad$ $\vec{v} = \langle \sin\theta, -\cos\theta, 0\rangle = \sin\theta\hat{i} - \cos\theta\hat{j}$

$$\cos\vartheta = \frac{\vec{u}\cdot\vec{v}}{\left\|\vec{u}\right\|\left\|\vec{v}\right\|} = \frac{\cos\theta\sin\theta - \sin\theta\cos\theta - 0}{\sqrt{\cos^2\theta+\sin^2\theta+1^2}\sqrt{\sin^2\theta+\cos^2\theta}} = \frac{0}{\sqrt{2}} = 0 \Rightarrow \vartheta = \arccos 0 = \tfrac{\pi}{2}$$

Perpendicular

- Example: # 79, §14.1

$\vec{u} = \langle 5,-4,3\rangle = 5\hat{i} - 4\hat{j} + 3\hat{k}$ $\qquad$ $\vec{v} = \langle 1,0,0\rangle = \hat{i}$

$$\mathrm{Proj}_v\vec{u} = \vec{u}_{\parallel} = \left(\frac{\vec{u}\cdot\vec{v}}{\left\|\vec{v}\right\|^2}\right)\vec{v} = \left(\frac{5-0+0}{1^2}\right)\hat{i} = 5\hat{i} = \langle 5,0,0\rangle$$

$$\therefore \vec{u}_{\perp} = \vec{u} - \vec{u}_{\parallel} = (5\hat{i} - 4\hat{j} + 3\hat{k}) - 5\hat{i} = -4\hat{j} + 3\hat{k} = \langle 0,-4,3\rangle$$

- Example: # 81, §14.1

$$\vec{u} = \langle 1,2,2 \rangle = \hat{i} + 2\hat{j} + 2\hat{k}$$

$$\cos\alpha = \frac{u_x}{\|u\|} = \frac{1}{\sqrt{1^2+2^2+2^2}} = \tfrac{1}{3}, \cos\beta = \frac{u_y}{\|u\|} = \frac{2}{\sqrt{1^2+2^2+2^2}} = \tfrac{2}{3},$$

$$\cos\gamma = \frac{u_z}{\|u\|} = \frac{2}{\sqrt{1^2+2^2+2^2}} = \tfrac{2}{3}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{1^2+2^2+2^2}{3^2} = 1$$

- <u>Example: # 13, §14.2</u>

$$\vec{u} = \langle -3,2,-5 \rangle = -3\hat{i} + 2\hat{j} - 5\hat{k} \qquad \vec{v} = \langle \tfrac{1}{2}, -\tfrac{3}{4}, \tfrac{1}{10} \rangle = \tfrac{1}{2}\hat{i} - \tfrac{3}{4}\hat{j} + \tfrac{1}{10}\hat{k}$$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = (u_x v_y - u_y v_x)\hat{k} - (u_x v_z - u_z v_x)\hat{j} + (u_y v_z - u_z v_y)\hat{i}$$

$$= \left(\tfrac{9}{4} - 1\right)\hat{k} - \left(-\tfrac{3}{10} + \tfrac{5}{2}\right)\hat{j} + \left(\tfrac{1}{5} - \tfrac{15}{4}\right)\hat{i} = -\tfrac{71}{20}\hat{i} - \tfrac{22}{10}\hat{j} + \tfrac{5}{4}\hat{k} = -\tfrac{71}{20}\hat{i} - \tfrac{11}{5}\hat{j} + \tfrac{5}{4}\hat{k} = \langle -\tfrac{71}{20}, -\tfrac{11}{5}, \tfrac{5}{4} \rangle$$

$$\vec{u}\cdot(\vec{u}+\vec{v}) = \left(-3\hat{i} + 2\hat{j} - 5\hat{k}\right)\cdot\left(-\tfrac{71}{20}\hat{i} - \tfrac{11}{5}\hat{j} + \tfrac{5}{4}\hat{k}\right) = \tfrac{213}{20} - \tfrac{22}{5} - \tfrac{25}{4} = \tfrac{213-88-125}{20} = 0$$

Hence $\vec{u}$ is orthogonal to $\vec{u}\times\vec{v}$

$$\vec{v}\cdot(\vec{u}+\vec{v}) = \left(\tfrac{1}{2}\hat{i} - \tfrac{3}{4}\hat{j} + \tfrac{1}{10}\hat{k}\right)\cdot\left(-\tfrac{71}{20}\hat{i} - \tfrac{11}{5}\hat{j} + \tfrac{5}{4}\hat{k}\right) = -\tfrac{71}{40} + \tfrac{33}{20} + \tfrac{5}{40} = \tfrac{-71+66+5}{40} = 0$$

Hence $\vec{v}$ is orthogonal to $\vec{u}\times\vec{v}$

- <u>Example: # 17, §14.2</u>

$$\vec{u} = \langle 3,2,-1 \rangle = 3\hat{i} + 2\hat{j} - \hat{k} \qquad \vec{v} = \langle 1,2,3 \rangle = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -1 \\ 1 & 2 & 3 \end{vmatrix} = (6-2)\hat{k} - (9+1)\hat{j} + (6+2)\hat{i} = 8\hat{i} + 10\hat{j} + 4\hat{k}$$

Area: $\|\vec{u}\times\vec{v}\| = \sqrt{8^2+10^2+4^2} = \sqrt{180} = \sqrt{36\cdot 5} = 6\sqrt{5}$