MA 262

SPRING 2008 KALLFELZ

CALCULUS II : NOTES FOR APRIL 3RD

Exercise 20, §12.4

$r = 3\cos 2\theta$ (4 petals)

· Tangents at origin: $r(\alpha) = 0 \Rightarrow \cos 2\alpha = 0 \Rightarrow 2\alpha_1 = \frac{\pi}{2} \rightarrow \alpha_1 = \frac{\pi}{4}$

$2\alpha_2 = \frac{3}{2}\pi \rightarrow \alpha_2 = \frac{3}{4}\pi$

$r'(\theta) = -6\sin 2\theta \rightarrow r'(\alpha_1) = -6 \neq 0$

$r'(\alpha_2) = 6 \neq 0$

· Extrema: $r'(\theta) = -6\sin 2\theta = 0 \Rightarrow \sin 2\theta = 0 \Rightarrow 2\theta_1 = 0 \rightarrow \theta_1 = 0$

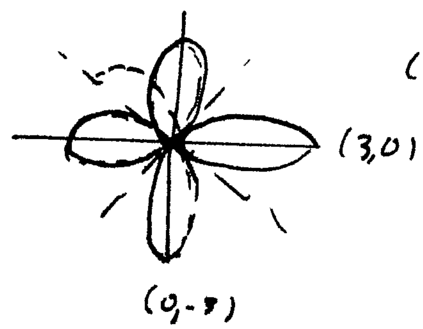
$2\theta_2 = \pi \rightarrow \theta_2 = \pi/2$

$r''(\theta) = -12\cos 2\theta \rightarrow r''(\theta_1) = -12 < 0 \therefore r(\theta_1) = 3$ max

$r''(\theta_2) = 12 > 0 \therefore r(\theta_2) = -3$ min

Sufficient to examine $0 < \theta < \pi/2$

θ	r
0	3
$\pi/8$	$\frac{3}{2}\sqrt{2}$
$\pi/6$	$\frac{3}{2}$
$\pi/4$	0
$\pi/3$	$-3/2$
$\pi/2$	-3



(Not to scale)

Exercise 18, §12.4

$r(\theta) = -\sin 5\theta$ (5 leaves)

$r'(\theta) = -5\cos 5\theta$

· Tangents at origin: $r(\theta) = 0 \Rightarrow \sin 5\alpha = 0 \Rightarrow 5\alpha = \sin^{-1}(0)$

$\Rightarrow \alpha_1 = 0$

$\alpha_2 = \pi/5$

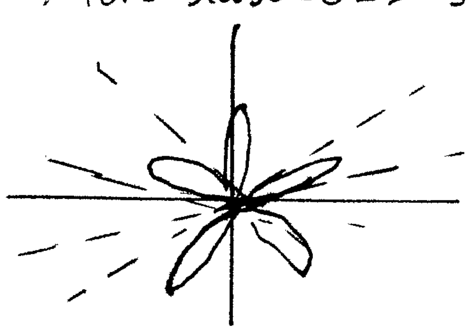
$\alpha_3 = \frac{2}{5}\pi$

$\alpha_4 = \frac{3}{5}\pi$

$\alpha_5 = \frac{4}{5}\pi$

$r'(\alpha_i) \neq 0$ $1 \leq i \leq 5$

$r'(\theta) = -5\cos 5\theta = 0 \Rightarrow 5\theta_1 = \pi/2 \Rightarrow \theta_1 = \pi/10$



26) $r = \dfrac{6}{2\sin\theta - 3\cos\theta} = \dfrac{6r}{2r\sin\theta - 3r\cos\theta} \Rightarrow r = \dfrac{6r}{2y-3x} \Rightarrow \boxed{2y-3x=6}$

(Line: $y = \frac{3}{2}x + 3$)

46) $r = 2\sin 2\theta$

(a) $\pi/6 \Rightarrow r'(\theta) = 2\sin 2(\theta + \pi/6) = 2[\sin 2\theta \cos \pi/3 + \cos 2\theta \sin \pi/3]$

$= 2[\sin 2\theta \cdot \frac{1}{2} + \frac{\sqrt{3}}{2}\cos 2\theta] = \sin 2\theta + \sqrt{3}\cos 2\theta$

(b) $\pi/2 \Rightarrow r(\theta) = 2\sin 2\theta \cos\pi + \cos 2\theta \sin\pi = -2\sin 2\theta$

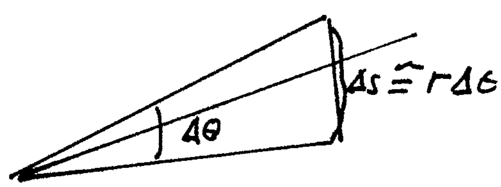
(c) $\frac{2}{3}\pi \Rightarrow r(\theta) = 2[\sin 2\theta \cos \frac{4}{3}\pi + \cos 2\theta \sin \frac{4}{3}\pi]$

$= 2[-\frac{1}{2}\sin 2\theta - \frac{\sqrt{3}}{2}\cos 2\theta] = -\sin 2\theta - \sqrt{3}\cos 2\theta$

(d) $\pi \Rightarrow r(\theta) = 2[\sin 2\theta \cos 2\pi + \cos 2\theta \sin 2\theta] = 2\sin 2\theta$

Integral Formulae:

Area $A = \displaystyle\int_\alpha^\beta \frac{1}{2}[r(\theta)]^2 d\theta$



$\therefore A = \dfrac{1}{2}\displaystyle\int_\alpha^\beta [r(\theta)]^2 d\theta = \lim_{\substack{N\to\infty \\ \Delta\theta_i \to 0}} \sum_{i=1}^{N} \frac{1}{2} r^2(\theta_i)\Delta\theta_i$

$\Delta A \approx \frac{1}{2}bh = \frac{1}{2}h\Delta s$

$\Delta s \approx r\Delta\theta \quad h \approx r$

$\therefore \Delta A \approx \frac{1}{2}r^2\Delta\theta \to dA = \frac{1}{2}r^2 d\theta$

Arclength $L = \displaystyle\int_a^b \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$ (For a general parametric representation)

$\dot{x} \to x'(\theta) = \dfrac{d}{d\theta}[r(\theta)\cos\theta] = r'(\theta)\cos\theta - r(\theta)\sin\theta$

$y'(\theta) = \dfrac{d}{dt}[r(\theta)\sin\theta] = r'(\theta)\sin\theta + r(\theta)\cos\theta$

$$\therefore \dot{x}^2+\dot{y}^2 = (r'(\theta)\cos\theta - r(\theta)\sin\theta)^2 + (r'(\theta)\sin\theta + r(\theta)\cos\theta)^2$$

$$= [r'(\theta)]^2\cos^2\theta - \cancel{2r'(\theta)r(\theta)\cos\theta\sin\theta} + r^2(\theta)\sin^2\theta + [r'(\theta)]^2\sin^2\theta$$
$$\cancel{+2r(\theta)r'(\theta)\sin\theta\cos\theta} + r^2(\theta)\cos^2\theta$$

$$= [r'(\theta)]^2(\cos^2\theta + \sin^2\theta) + r^2(\theta)[\sin^2\theta + \cos^2\theta] = [r'(\theta)]^2 + r^2(\theta)$$

$$\therefore \quad L = \int_\alpha^\beta \sqrt{r(\theta)^2 + (r'(\theta))^2}\, d\theta$$

$$S_x = 2\pi\int_\alpha^\beta y\,ds = 2\pi\int_\alpha^\beta r(\theta)\sin\theta\sqrt{r(\theta)^2+(r'(\theta))^2}\,d\theta$$

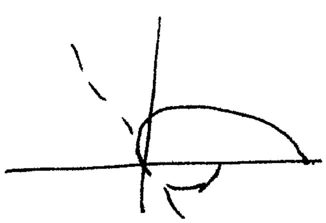
$$S_y = 2\pi\int_\alpha^\beta x\,ds = 2\pi\int_\alpha^\beta r(\theta)\cos\theta\sqrt{r(\theta)^2+(r'(\theta))^2}\,d\theta$$

Surfaces of Revolution

• Example # 9, § 12.5

(Inner Loop) $\quad r(\theta) = 1 + 2\cos\theta$

$$r(\alpha) = 0 \Rightarrow 0 = 1 + 2\cos\theta \Rightarrow \cos\theta = -\tfrac{1}{2} \Rightarrow \theta_1 = \tfrac{2}{3}\pi$$



$$\therefore A = \frac{1}{2}\int_{2/3\pi}^{4/3\pi} r^2(\theta)\,d\theta = \int_{2/3\pi}^{\pi} r^2(\theta)\,d\theta$$

$$= \int_{2/3\pi}^{\pi} (1+2\cos\theta)^2\,d\theta$$

$$= \int_{2/3\pi}^{\pi} (1 + 4\cos\theta + 4\cos^2\theta)\,d\theta$$

$$= \int_{2/3\pi}^{\pi} (1 + 4\cos\theta + 2 + 2\cos 2\theta)\,d\theta$$

$$= [3\theta + 4\sin\theta + \sin 2\theta]\Big|_{2/3\pi}^{\pi} = 3\cdot\frac{\pi}{3} - 4\sin\frac{2}{3}\pi - \sin\frac{4}{3}\pi$$

$= \pi - 2\sqrt{3} + \frac{\sqrt{3}}{2} = \pi + \frac{\sqrt{3} - 4\sqrt{3}}{2} = \pi - \frac{3}{2}\sqrt{3} = \frac{1}{2}(2\pi - 3\sqrt{3}) = A_2$

Area between loops

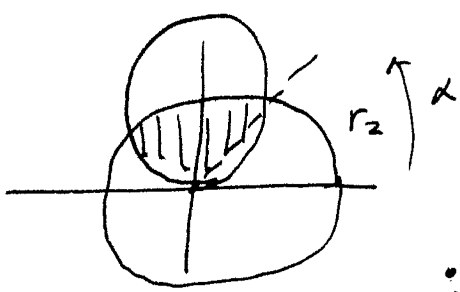
Area of outer loop: $2 \cdot \frac{1}{2}\int_0^{2/3\pi} r^2(\theta) = [3\theta + 4\sin\theta + \sin 2\theta]\Big|_0^{2/3\pi}$

$A_1 = \frac{6}{3}\pi + 4\sin\frac{2}{3}\pi + \sin\frac{4}{3}\pi = 2\pi + 2\sqrt{3} - \frac{\sqrt{3}}{2} = 2\pi + \frac{3}{2}\sqrt{3}$

$\therefore$ Between Loops: $A_1 - A_2 = (2\pi + \frac{3}{2}\sqrt{3}) - \frac{1}{2}(2\pi - 3\sqrt{3})$

$= \pi + 3\sqrt{3}$

Example # 28: $r_1 = 4\sin\theta \qquad r_2 = 2$



$r_1 = 4\sin\alpha = r_2 = 2$

$\Rightarrow \sin\alpha = 1/2 \Rightarrow \alpha = \pi/6$

$\therefore A = 2\left\{\frac{1}{2}\int_0^{\pi/6} r_1^2\, d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} r_2^2\, d\theta\right\}$

$= \int_0^{\pi/6} 16\sin^2\theta\, d\theta + 4\int_{\pi/6}^{\pi/2} d\theta$

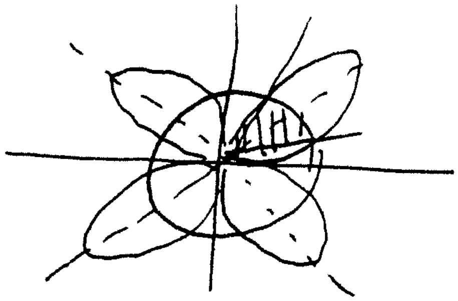
$= 8\int_0^{\pi/6}[1 - \cos 2\theta]\, d\theta + 4\theta\Big|_{\pi/6}^{\pi/2}$

$= [8\theta - 4\sin 2\theta]\Big|_0^{\pi/6} + 4 \cdot \pi/3$

$= \frac{4}{3}\pi - 4\sin\frac{\pi}{3} + \frac{4}{3}\pi = \frac{8}{3}\pi - 2\sqrt{3} = \frac{2}{3}(4\pi - 3\sqrt{3})$

Example #25

$r_1 = 4\sin 2\theta \qquad r_2 = 2$



$$A_1 = 2\cdot\left\{\frac{1}{2}\int_0^{\alpha} r_1^2\,d\theta + \frac{1}{2}\int_{\alpha}^{\pi/4} r_2^2\,d\theta\right\}$$

$$\Rightarrow\ 4\sin 2\alpha = 2 \Rightarrow 2\alpha = \frac{\pi}{6} \Rightarrow \alpha = \frac{\pi}{12}$$

$$A_1 = 16\int_0^{\pi/12}\sin^2 2\theta\,d\theta + 4\int_{\pi/12}^{\pi/4} d\theta$$

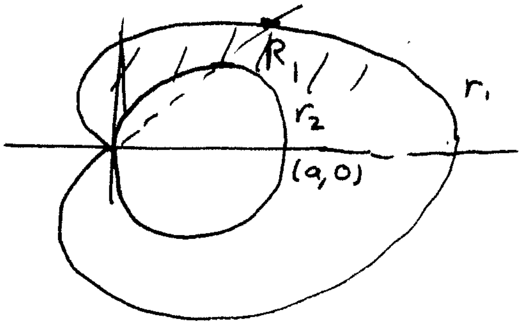
$$= 8\int_0^{\pi/12}[1 - \cos 4\theta]\,d\theta + 4\theta\Big|_{\pi/12}^{\pi/4}$$

$$= [8\theta - 2\sin 4\theta]\Big|_0^{\pi/12} + 4\pi/6$$

$$= \frac{2}{3}\pi - 2\sin\frac{\pi}{3} + 2\frac{\pi}{3} = \frac{4}{3}\pi - \sqrt{3} = \frac{1}{3}(4\pi - 3\sqrt{3})$$

There are 4 such regions, so total area $= 4A_1 = \frac{4}{3}(4\pi - 3\sqrt{3})$

Example #31 Inside $r_1 = a(1+\cos\theta)$ outside $r_2 = a\cos\theta$



Method 1: $A = 2\cdot\frac{1}{2}\int_0^{\pi/2}[r_1^2 - r_2^2]\,d\theta + 2\cdot\frac{1}{2}\int_{\pi/2}^{\pi} r_1^2\,d\theta$

Note ~~that~~ (from figure $R_1 : 0 \le \theta \le \frac{\pi}{2}$ where $r_2 > 0$

r_2 not present in 2nd quadr.

$$= a^2\int_0^{\pi/2}[1 + 2\cos\theta + \cos^2\theta - \cos^2\theta]\,d\theta$$

$$+ a^2\int_{\pi/2}^{\pi}(1 + 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= a^2\left\{[\theta + 2\sin\theta]\Big|_0^{\pi/2} + [\theta + 2\sin\theta]\Big|_{\pi/2}^{\pi} + \frac{1}{2}\int_{\pi/2}^{\pi}(1 + \cos 2\theta)\,d\theta\right\}$$

$$= a^2\left[\pi/2 + \cancel{2} + \pi/2 - \cancel{2} + \frac{1}{2}\left[\theta + \frac{1}{2}\sin\theta\right]\Big|_{\pi/2}^{\pi}\right]$$

$$= a^2\left[\pi + \frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{5}{4}\pi a^2$$

Method 2: The area of the inscribed circle $= \pi r^2 = \pi\left(\frac{a}{2}\right)^2 = \frac{\pi a^2}{4}$

Hence $A = \int_0^{\pi} r_1^2\, d\theta - \frac{\pi}{4}a^2 = a^2 \int_0^{\pi} [1+\cos\theta]^2 d\theta - \frac{\pi}{4}a^2$

$= a^2 \int_0^{\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta - \frac{\pi}{4}a^2$

$= a^2 \left\{ [\theta + 2\sin\theta]_0^{\pi} + \frac{1}{2}\int_0^{\pi} (1+\cos 2\theta)\, d\theta \right\} - \frac{\pi}{4}a^2$

$= a^2 \left\{ \pi + [\frac{\theta}{2} + \frac{1}{2}\sin 2\theta]\Big|_0^{\pi} \right\} - \frac{\pi}{4}a^2$

$= \frac{3}{2}\pi a^2 - \frac{\pi}{4}a^2 = \frac{5}{4}\pi a^2$

Exercise 38:

$L = \int_0^{\pi/3} \sqrt{r^2 + r'^2}\, d\theta$

$r(\theta) = \sec\theta$

$r'(\theta) = \sec\theta\tan\theta$

$= \int_0^{\pi/3} \sqrt{\sec^2\theta + \sec^2\theta\tan^2\theta}\, d\theta = \int_0^{\pi/3} \sec\theta\sqrt{1+\tan^2\theta}\, d\theta$

$= \int_0^{\pi/3} \sec\theta\cdot\sec\theta\, d\theta = \int_0^{\pi/3} \sec^2\theta\, d\theta = \tan\theta\Big|_0^{\pi/3} = \tan\pi/3 = \sqrt{3}$

Exercise 44

$S_y = 2\pi \int_0^{\pi/2} r(\theta)\cos\theta \sqrt{r^2 + r'^2}\, d\theta \qquad r = a\cos\theta$

$r'(\theta) = -a\sin\theta$

$= 2\pi \int_0^{\pi/2} (a\cos\theta)\cos\theta\sqrt{a^2\cos^2\theta + a^2\sin^2\theta} = 2\pi a^2 \int_0^{\pi/2} \cos^2\theta\, d\theta$

$= \pi a^2 \int_0^{\pi/2} (1+\cos 2\theta)\, d\theta = \pi a^2 \left\{\theta + \frac{1}{2}\sin 2\theta\right\}\Big|_0^{\pi/2} = \frac{\pi^2 a^2}{2}$