MA262 **Notes for April 29**
Spring 2008 Kallfelz

- *REVIEW PROBLEMS-FINAL EXAM (CHOOSE 5 FROM 6 + 1 BONUS)*

I.a) Evaluate $\int \sqrt{1+\sqrt{x}}\,dx$

$$u = 1 + x^{1/2} \Rightarrow du = \tfrac{1}{2}x^{-1/2}dx \Rightarrow dx = 2x^{1/2}du = 2(u-1)du$$
$$\therefore \int \sqrt{1+x^{1/2}}\,dx = \int \sqrt{u}\,(2(u-1))du = \int (2u^{3/2} - u^{1/2})du = \tfrac{4}{5}u^{5/2} - \tfrac{2}{3}u^{3/2} + C$$
$$= 2u^{3/2}\left(\tfrac{2}{5}u - \tfrac{1}{3}\right) + C = 2\left(1+\sqrt{x}\right)^{3/2}\left(\tfrac{2}{5} + \tfrac{2}{5}\sqrt{x} - \tfrac{1}{3}\right) + C = 2\left(1+\sqrt{x}\right)\left(\tfrac{2}{5}\sqrt{x} - \tfrac{1}{15}\right) + C$$
$$= \tfrac{2}{15}\left(1+\sqrt{x}\right)\left(6\sqrt{x} - 1\right) + C$$

b.) Evaluate $\int \frac{dx}{x\sqrt{1+x^2}}$ using a trigonometric substitution

$$x(\theta) = \tan\theta \to dx = \sec^2\theta d\theta$$
$$\int \frac{dx}{x\sqrt{1+x^2}} = \int \frac{\sec^2\theta d\theta}{\tan\theta\sec\theta} = \int \frac{\sec\theta}{\tan\theta}d\theta = \int \csc\theta d\theta = \ln\left|\csc\theta - \cot\theta\right| + C$$
$$\tan\theta = x = \frac{OPP}{ADJ} \to HYP = \sqrt{x^2+1} \to \cot\theta = \tfrac{1}{x}, \csc\theta = \frac{\sqrt{1+x^2}}{x}$$
$$= \ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + C = \ln\left|\frac{\sqrt{1+x^2}-1}{x}\right| + C$$

c.) Repeat b.) with substitution: $u^2 = 1 + x^2 \Rightarrow 2udu = 2xdx \Rightarrow dx = \frac{u}{x}du$

$$\int \frac{dx}{x\sqrt{1+x^2}} = \int \frac{u}{x^2u}du = \int \frac{du}{u^2-1} \Rightarrow \frac{1}{(u-1)(u+1)} = \frac{A_1}{(u-1)} + \frac{A_2}{(u+1)} \Rightarrow 1 = A_1(u+1) + A_2(u-1)$$
$$A_1 = \tfrac{1}{2}, A_2 = -\tfrac{1}{2} \Rightarrow \int \frac{du}{u^2-1} = \tfrac{1}{2}\int\left[\frac{du}{u-1} - \frac{du}{u+1}\right] = \tfrac{1}{2}\ln|u-1| - \tfrac{1}{2}\ln|u+1| = \tfrac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C$$
$$= \tfrac{1}{2}\ln\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right| + C = \tfrac{1}{2}\ln\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\cdot\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}-1}\right| + C = \tfrac{1}{2}\ln\left|\frac{x^2+1-2\sqrt{x^2+1}+1}{x^2+1-1}\right| + C$$
$$= \tfrac{1}{2}\ln\left|\frac{\left(\sqrt{x^2+1}-1\right)^2}{x^2}\right| + C = \ln\sqrt{\frac{\left(\sqrt{x^2+1}-1\right)^2}{x^2}} + C = \ln\left|\frac{\sqrt{x^2+1}-1}{x}\right| + C$$

II. Given that $\int_0^\infty x^n e^{-x}dx$ converges, for all $n > 0$, show that $\int_0^\infty e^{-x^2}dx$ converges

$$u = x^2 \Rightarrow du = 2xdx \Rightarrow dx = \tfrac{1}{2}\frac{du}{x} = \tfrac{1}{2}u^{-1/2}du$$

$$\therefore \int_0^\infty e^{-x^2}dx = \tfrac{1}{2}\int_0^\infty u^{-1/2}e^{-u}du$$

$$U = e^{-u} \rightarrow dU = -e^{-u}du, dV = u^{-1/2}du \rightarrow V = 2u^{1/2}$$

$$\therefore \tfrac{1}{2}\int_0^{\Re} u^{-1/2}e^{-u}du = \tfrac{1}{2}\left\{2u^{1/2}e^{-u}\Big|_0^\infty + 2\int_0^\infty u^{1/2}e^{-u}du\right\}$$

The integral on the right converges by virtue of the above statement

Examine convergence of first term:

$$u^{1/2}e^{-u}\Big|_0^\infty = \lim_{b\to\infty}\frac{\sqrt{u}}{e^u}\Bigg|_0^b = \lim_{b\to\infty}\frac{b^{1/2}}{e^b}\xrightarrow{LHR}\lim_{b\to\infty}\frac{\tfrac{1}{2}b^{-1/2}}{e^b} = \tfrac{1}{2}\lim_{b\to\infty}\frac{1}{\sqrt{b}e^b} = 0$$

Hence $\int_0^\infty e^{-x^2}dx$ converges.

III.a) Test convergence for: $\sum_{k=1}^\infty \frac{k!}{e^k}$

Using **RaT:** $\rho = \lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\frac{(k+1)!}{e^{k+1}}\cdot\frac{e^k}{k!} = \frac{1}{e}\lim_{k\to\infty}(k+1) = \infty$ diverges

b) Test convergence for: $\sum_{k=1}^\infty \frac{1}{\sqrt{k^3+2k}}$

<u>Method 1:</u> According to **SCT** : $k^3 + 2k > k^3$, for all $k > 0$

$\therefore \sum\frac{1}{k^{3/2}} > \sum\frac{1}{\sqrt{k^3+2k}}$ But $\sum\frac{1}{k^{3/2}}$ is a convergent *p*-series. Hence $\sum_{k=1}^\infty\frac{1}{\sqrt{k^3+2k}}$ converges

Method 2: Using **LCT:**

$$\lim_{k\to\infty}\frac{k^{3/2}}{\sqrt{k^3+2k}}=\lim_{k\to\infty}\sqrt{\frac{k^3}{k^3+2k}}=\lim_{k\to\infty}\sqrt{\frac{1}{1+\frac{2}{k^2}}}=1$$

Hence: $0<\lim_{k\to\infty}\frac{k^{3/2}}{\sqrt{k^3+2k}}<\infty$**,** so $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k^3+2k}}$ converges, since $\sum\frac{1}{k^{3/2}}$ is a convergent p-series.

c.) Find interval of convergence for the power series: $\sum_{k=1}^{\infty}(-1)^k\frac{(x+1)^k}{(k+1)^2}$

To find interval of absolute convergence, use **RaT:**

$$\rho(x)=\lim_{k\to\infty}\frac{|a_{k+1}(x)|}{|a_k(x)|}=\lim_{k\to\infty}\frac{|x+1|^{k+1}}{(k+2)^2}\cdot\frac{(k+1)^2}{|x+1|^k}=|x+1|\lim_{k\to\infty}\left(\frac{k+1}{k+2}\right)^2$$

$$=|x+1|\lim_{k\to\infty}\left(\frac{1+\frac{1}{k}}{1+\frac{2}{k}}\right)^2=|x+1|<1\Rightarrow -1<x+1<1\Rightarrow -2<x<0\Rightarrow I_{-1}^{1}=(-2,0)$$

Testing endpoints: x = -2: $\sum_{k=1}^{\infty}(-1)^k\frac{(-2+1)^k}{(k+1)^2}=\sum_{k=1}^{\infty}\frac{(-1)^{2k}}{(k+1)^2}=\sum_{k=1}^{\infty}\frac{1}{(k+1)^2}=\sum_{j=2}^{\infty}\frac{1}{j^2}$ a convergent p series. x = 0: $\sum_{k=1}^{\infty}(-1)^k\frac{(0+1)^k}{(k+1)^2}=\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)^2}$, an absolutely convergent alternating series (as demonstrated in the x = -2 case). Hence interval of convergence is:

$$[-2,0]=\{x\mid -2\le x\le 0\}$$

- **IV.a)** Find the Taylor Series representation for $f(x)=\sin x$ (i.e. its power-series representation centered at any point c) from its McCluarin Series $f(x)=\sin x=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}$.

Using the 'trick of 0':

$$\sin x=\sin(x+c-c)=\sin(c+(x-c))=\sin c\cos(x-c)+\cos c\sin(x-c)$$
$$=\sin c\cos u+\cos c\sin u$$

(where u = (x – c))

Hence inserting the McClaurin Series:

$$\begin{aligned}
\sin x &= \sin c\cos u + \cos c\sin u = \sin c\sum_{k=0}^{\infty}\tfrac{(-1)^k}{(2k)!}u^{2k} + \cos c\sum_{k=0}^{\infty}\tfrac{(-1)^k}{(2k+1)!}u^{2k+1} \\
&= \sin c\sum_{k=0}^{\infty}\tfrac{(-1)^k}{(2k)!}(x-c)^{2k} + \cos c\sum_{k=0}^{\infty}\tfrac{(-1)^k}{(2k+1)!}(x-c)^{2k+1} \\
&= \sum_{k=0}^{\infty}(-1)^k\left[\tfrac{\sin c}{(2k)!}(x-c)^{2k} + \tfrac{\cos c}{(2k+1)!}(x-c)^{2k+1}\right] \\
&= \sum_{k=0}^{\infty}(-1)^k\left[\tfrac{\sin c}{(2k)!} + \tfrac{\cos c}{(2k+1)!}(x-c)\right](x-c)^{2k}
\end{aligned}$$

b.) Show that the McCluarin Series is recovered in the $c \to 0$ limit

$$\sum_{k=0}^{\infty}(-1)^k\left[\tfrac{\sin 0}{(2k)!} + \tfrac{\cos 0}{(2k+1)!}(x-0)\right](x-0)^{2k} = \sum_{k=0}^{\infty}(-1)^k\left[0 + \tfrac{x}{(2k+1)!}\right]x^{2k} = \sum_{k=0}^{\infty}(-1)^k\,\tfrac{x^{2k+1}}{(2k+1)!}$$

c.) Find the McCluarin series for $f(x) = \int_0^x \sin^2 t\,dt$

$$\begin{aligned}
f(x) &= \int_0^x \sin^2 t\,dt = \tfrac{1}{2}\int_0^x (1-\cos 2t)dt = \tfrac{1}{2}x - \tfrac{1}{2}\int_0^x\sum_{k=0}^{\infty}(-1)^k\frac{(2t)^{2k}}{2k!}dt \\
&= \tfrac{1}{2}x - \tfrac{1}{2}\sum_{k=0}^{\infty}(-1)^k\tfrac{2^{2k}}{2k!}\int_0^x t^{2k}dt = \tfrac{1}{2}x - \tfrac{1}{2}\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k}x^{2k+1}}{(2k+1)(2k!)} = \tfrac{1}{2}x - \sum_{k=0}^{\infty}\frac{(-1)^k 2^{2k-1}x^{2k+1}}{(2k+1)(2k!)}
\end{aligned}$$

V.a.) Convert to rectangular coordinates: $r = 4\cos 2\theta\sec\theta$

$$\begin{aligned}
& r = 4\cos 2\theta\sec\theta = 4\left(\cos^2\theta - \sin^2\theta\right)\sec\theta = 4\cos\theta - \frac{4\sin^2\theta}{\cos\theta} \\
& \Rightarrow r^2 = 4r\cos\theta - \frac{4r\sin^2\theta}{\cos\theta} \Rightarrow r^2 = 4r\cos\theta - \frac{4r^2\sin^2\theta}{r\cos\theta} \Rightarrow x^2 + y^2 = 4x - \frac{4y^2}{x} \\
& \Rightarrow x^3 + xy^3 = 4\left(x^2 - y^2\right)
\end{aligned}$$

b.) Describe conic and convert to polar coordinates:

$$x^2 - 2x + 2y^2 = 0 \to x^2 - 2x + 1 + 2y^2 = 1 \to (x-1)^2 + \frac{y^2}{1/2} = 1 \Rightarrow \frac{(x-1)^2}{1^2} + \frac{(y-0)^2}{\left(\frac{\sqrt{2}}{2}\right)^2} = 1$$

Ellipse, centered at (1, 0) with semimajor length: $a = 1$, and semiminor length $b = \frac{\sqrt{2}}{2}$

c.) Find the area shared in common with $r_1 = 3, r_2 = 6\cos\theta$

The first is a circle centered at the origin with radius 3
The second is a circle of the same radius centered at (3,0)

As shown in class,

$$\iint_{\Omega} r\,dr\,d\theta = 2\left\{3^2\tfrac{\pi}{3} + \int_{\pi/3}^{\pi/2}\int_0^{6\cos\theta} r\,dr\,d\theta\right\} = 6\pi + \int_{\pi/3}^{\pi/2} r^2\Big|_0^{6\cos\theta} d\theta = 6\pi + 36\int_{\pi/3}^{\pi/2}\cos^2\theta d\theta$$

$$= 6\pi + 18\int_{\pi/3}^{\pi/2}(1+\cos 2\theta)d\theta = 6\pi + 18\left(\tfrac{\pi}{6}\right) + 9\sin 2\theta\Big|_{\pi/3}^{\pi/2} = 9\pi - 9\sin\tfrac{2}{3}\pi = 9\left(\pi - \tfrac{\sqrt{3}}{2}\right)$$

VI.) a.) Given $\vec{f}(t) = t^2\hat{i} - \cos t\hat{j} + \sin t\hat{k}$ $\quad \vec{g}(t) = e^t\hat{i} - \ln t\hat{k}$ find the derivatives of their dot products and cross-products:

<u>Method 1</u>: $\frac{d}{dt}\left(\vec{f}\cdot\vec{g}\right) = \dfrac{d}{dt}\left(t^2e^t - \ln t\sin t\right) = 2te^t + t^2e^t - \dfrac{1}{t}\sin t - \ln t\cos t$

<u>Method 2:</u>

$$\tfrac{d}{dt}\left(\vec{f}\cdot\vec{g}\right) = \left(\vec{f}'\cdot\vec{g}\right) + \left(\vec{f}\cdot\vec{g}'\right) = \left(2t\hat{i} + \sin t\hat{j} + \cos t\hat{k}\right)\cdot\left(e^t\hat{i} - \ln t\hat{k}\right) + \left(t^2\hat{i} - \cos t\hat{j} + \sin t\hat{k}\right)\cdot\left(e^t\hat{i} - \tfrac{1}{t}\hat{k}\right)$$

$$= 2te^t - \ln t\cos t + t^2e^t - \frac{1}{t}\sin t$$

<u>Method 1:</u>

$$\tfrac{d}{dt}\left(\vec{f}\times\vec{g}\right) = \frac{d}{dt}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ t^2 & -\cos t & \sin t\\ e^t & 0 & -\ln t\end{vmatrix} = \frac{d}{dt}\left[\hat{i}(\ln t\cos t) - \hat{j}\left(-t^2\ln t - e^t\sin t\right) + \hat{k}\left(e^t\cos t\right)\right]$$

$$= \hat{i}\left(\tfrac{1}{t}\cos t - \ln t\sin t\right) - \hat{j}\left(-t - 2t\ln t - e^t\sin t - e^t\cos t\right) + \hat{k}\left(e^t\cos t - e^t\sin t\right)$$

<u>Method 2:</u>

$$\tfrac{d}{dt}\left(\vec{f}\cdot\vec{g}\right) = \left(\vec{f}'\times\vec{g}\right) + \left(\vec{f}\times\vec{g}'\right) = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2t & \sin t & \cos t\\ e^t & 0 & -\ln t\end{vmatrix} + \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ t^2 & -\cos t & \sin t\\ e^t & 0 & -\tfrac{1}{t}\end{vmatrix}$$

$$\left[\hat{i}(-\ln t\sin t) - \hat{j}\left(-2t\ln t - e^t\cos t\right) + \hat{k}\left(-e^t\sin t\right)\right] + \left[\hat{i}\left(\tfrac{1}{t}\cos t\right) - \hat{j}\left(-t - e^t\sin t\right) + \hat{k}\left(e^t\cos t\right)\right]$$

$$= \hat{i}\left(\tfrac{1}{t}\cos t - \ln t\sin t\right) - \hat{j}\left(-t - 2t\ln t - e^t\sin t - e^t\cos t\right) + \hat{k}\left(e^t\cos t - e^t\sin t\right)$$

V.b.) Given $f(x,y,z) = x^2 + y^2 + z^2 - 2zx - 2y$, find $\vec{\nabla}f$

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = (2x-2z)\hat{i} + (2y-2)\hat{j} + (2z-2x)\hat{k} = 2\left[(x-z)\hat{i} + (y-1)\hat{j} + (z-x)\hat{k}\right]$$

c) Given your answer in b.), find the directional derivative at (2,1,0) in the direction $\hat{u} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$, for the angles $\alpha = \frac{\pi}{2}, \beta = \gamma = \frac{\pi}{4}$

$$D_u f(2,1,0) = \vec{\nabla} f(2,1,0) \cdot \hat{u} = \left(4\hat{i} - 4\hat{k}\right) \cdot \left(\tfrac{\sqrt{2}}{2}\hat{i} + \tfrac{\sqrt{2}}{2}\hat{k}\right) = 0$$

BONUS: A change of variables (to polar coordinates) double integral problem