MA262 **Notes for April 24**
Spring 2008 Kallfelz

- *THE DOUBLE INTEGRAL (Cont.) GEOMETRIC INTERPRETATION*

As mentioned in the previous notes[1] as well as in §16.2 (text): Just as the definite integral for a $f : R \to R$ mapping in a closed interval [*a* , *b*] defines an *area* under the graph of *y* = *f*(*x*) (for all *x* such that *f*(*x*) ≥ 0)[2], so the definite integral for a $f : R^2 \to R$ mapping in a simply connected region[3] Ω defines a *volume* under the graph of *z* = *f*(*x, y*). This point is brought analytically to home in the definition of the double integral as a limit of a Riemannian sum (in exact analogy with that of the usual integral)

$$\iint_{\Omega} f(x, y)dxdy = \iint_{\Omega} f(x, y)dydx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{N} f(x_i, y_i)\Delta x_i \Delta y_i = \lim_{N \to \infty, \Omega\_fixed} \sum_{i=1}^{N} f(x_i, y_i)\Delta x_i \Delta y_i$$

- **Note 1**: Observe how Fubini's Theorem is invoked in the first two terms on the left hand side.
- **Note 2**: The last term on the right hand side is another way of expressing an infinitely fine partition of Ω: Hold Ω fixed and shrink the width of each rectangle$_i$ in region Ω with area$_i$ = $\Delta x_i \Delta y_i$ to arbitrarily small size, which of course is effected by allowing the number of rectangles $N \to \infty$ subject to the aforementioned constraint.

- Example (# 3, §16.2 modified)

Consider the function $f(x, y) = x^2 + y^2$. Approximate the volume under *f* in the region $\Omega = \{(x, y) | 0 \le x \le 2, 0 \le y \le 4\}$ through the following successively finer even[4] partitions:

$$\mathscr{P}_1 = \{\Delta_i \,|\, i = 1,\ldots,8 = 4^0 \cdot 8 \,\&\, \Delta x_i \Delta y_j = 1 = (\tfrac{1}{2})^0\}$$
$$\mathscr{P}_2 = \{\Delta_i \,|\, i = 1,\ldots,32 = 4^1 \cdot 8 \,\&\, \Delta x_i \Delta y_j = \tfrac{1}{4} = (\tfrac{1}{2})^2 = \tfrac{1}{2^2} = 2^{-2\cdot 1}\}$$
$$\vdots$$
$$\mathscr{P}_{\mathrm{n}} = \{\Delta_i \,|\, i = 1,\ldots,128 = 4^{(n-1)} \cdot 8 \,\&\, \Delta x_i \Delta y_j = \tfrac{1}{16} = 2^{-2\cdot(n-1)}\}$$

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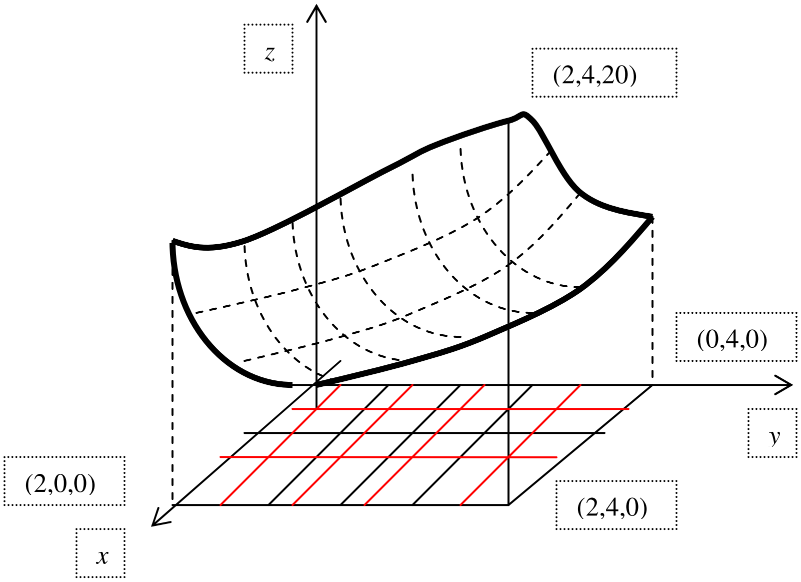
[1] Pages 2-4, April 22 notes, http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Apr22.pdf

[2] And for the opposite case, i.e. for all *x* in [*a* , *b*] such that *f*(*x*) < 0, then the integral produces the negative of the area under *f*'s curve. In this respect, as discussed in pp. 10-17 in the October 1 notes: http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Oct1notes.pdf
…it's misleading to identify the concept of definite integral with that of the *area*. The former is more general than the latter. It's best therefore to think of a definite integral as representing an *amount.* It makes sense to speak negative amounts: consider having a negative balance in your bank account (hopefully not very often) or consider a "negative" amount of dirt in a hole dug into the ground.

[3] Recall the definition of a simply connected region, as discussed in footnote 1, p. 3 April 22 notes http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Apr22.pdf

[4] I.e., partitions of into squares of equal area

This is suggested by the illustration below:



The red lines represent the where the finer partition $\mathscr{P}_2$ occurs outside the bounds of the coarser partition $\mathscr{P}_1$ (etched in black)

Hence:

$$A_1 = \sum_{1=1}^{N=8} f(x_i, y_i)\Delta x_i \Delta y_j$$
$$= \tfrac{1}{2^0}\left\{\left(\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2\right) + \left(\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{3}{2}\right)^2\right) + \ldots + \left(\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{7}{2}\right)^2\right) + \left(\left(\tfrac{3}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2\right) + \left(\left(\tfrac{3}{2}\right)^2 + \left(\tfrac{3}{2}\right)^2\right) + \ldots + \left(\left(\tfrac{3}{2}\right)^2 + \left(\tfrac{7}{2}\right)^2\right)\right\}$$

$$A_2 = \sum_{1=1}^{N=32} f(x_i, y_i)\Delta x_i \Delta y_j$$
$$= \tfrac{1}{2^2}\left\{\left(\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{4}\right)^2\right) + \left(\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{3}{4}\right)^2\right) + \ldots + \left(\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{15}{4}\right)^2\right) + \ldots + \left(\left(\tfrac{7}{4}\right)^2 + \left(\tfrac{1}{4}\right)^2\right) + \left(\left(\tfrac{7}{4}\right)^2 + \left(\tfrac{3}{2}\right)^2\right) + \ldots + \left(\left(\tfrac{7}{4}\right)^2 + \left(\tfrac{15}{4}\right)^2\right)\right\}$$

$$\vdots$$

$$A_n = \sum_{1=1}^{N=4^{n-1}\cdot 8} f(x_i, y_i)\Delta x_i \Delta y_j$$
$$= \tfrac{1}{2^{2(n-1)}}\left\{\left(\left(\tfrac{1}{2^n}\right)^2 + \left(\tfrac{1}{2^n}\right)^2\right) + \left(\left(\tfrac{1}{2^n}\right)^2 + \left(\tfrac{1+2^{n-1}}{2^n}\right)^2\right) + \ldots + \left(\left(\tfrac{1}{2^n}\right)^2 + \left(\tfrac{4\cdot 2^n-1}{2^n}\right)^2\right) + \ldots + \left(\left(\tfrac{2\cdot 2^n-1}{2^n}\right)^2 + \left(\tfrac{1}{2^n}\right)^2\right) + \ldots + \left(\left(\tfrac{2\cdot 2^n-1}{2^n}\right)^2 + \left(\tfrac{4\cdot 2^n-1}{2^n}\right)^2\right)\right\}$$

The first two results are summarized below (using Excel):

A1	**N**	**xi**	**yi**	**f(xi,yi)**
	1	0.5	0.5	0.5
	2	0.5	1.5	2.5
	3	0.5	2.5	6.5
	4	0.5	3.5	12.5
	5	1.5	0.5	2.5
	6	1.5	1.5	4.5
	7	1.5	2.5	8.5
	8	1.5	3.5	14.5
			Sum	**52**
A2	1	0.25	0.25	0.125
	2	0.25	0.75	0.625
	3	0.25	1.25	1.625
	4	0.25	1.75	3.125
	5	0.25	2.25	5.125
	6	0.25	2.75	7.625
	7	0.25	3.25	10.625
	8	0.25	3.75	14.125
	9	0.75	0.25	0.625
	10	0.75	0.75	1.125
	11	0.75	1.25	2.125
	12	0.75	1.75	3.625
	13	0.75	2.25	5.625
	14	0.75	2.75	8.125
	15	0.75	3.25	11.125
	16	0.75	3.75	14.625
	17	1.25	0.25	1.625
	18	1.25	0.75	2.125
	19	1.25	1.25	3.125
	20	1.25	1.75	4.625
	21	1.25	2.25	6.625
	22	1.25	2.75	9.125
	23	1.25	3.25	12.125
	24	1.25	3.75	15.625
	25	1.75	0.25	3.125
	26	1.75	0.75	3.625
	27	1.75	1.25	4.625
	28	1.75	1.75	6.125
	29	1.75	2.25	8.125
	30	1.75	2.75	10.625
	31	1.75	3.25	13.625
	32	1.75	3.75	17.125
			Sum	**53**

…where the “sum” is computed via the above Riemannian expression.

Evaluating the exact answer:

$$\int_0^4\int_0^2\left(x^2+y^2\right)dxdy=\int_0^4\left[\int_o^2\left(x^2+y^2\right)dx\right]dy=\int_0^4\left[\tfrac{1}{3}x^3+y^2x\right]_0^2dy=\int_0^4\left(\tfrac{8}{3}+2y^2\right)dy$$
$$=\left[\tfrac{8}{3}y+\tfrac{2}{3}y^3\right]_0^4=\tfrac{2}{3}y\left(4+y^2\right)\Big|_0^4=\tfrac{8}{3}(20)=\tfrac{160}{3}=53.\overline{3}$$

Which of course (as established by Fubini’s Theorem) can also be alternatively evaluated

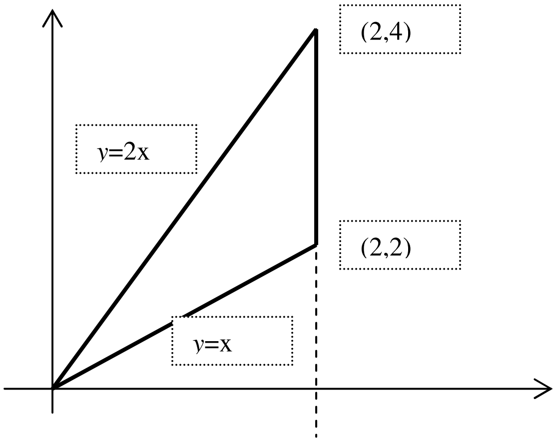
$$\int_0^2\int_0^4\left(x^2+y^2\right)dydx=\int_0^2\left[\int_o^4\left(x^2+y^2\right)dy\right]dx=\int_0^2\left[x^2y+\tfrac{1}{3}y^3\right]_0^4dx=\int_0^2\left(4x^2+\tfrac{64}{3}\right)dx$$
$$=\left[\tfrac{4}{3}x^3+\tfrac{64}{3}x\right]_0^2=\tfrac{4}{3}x\left(x^2+16\right)\Big|_0^2=\tfrac{8}{3}(20)=\tfrac{160}{3}=53.\overline{3}$$

Note how the above two partitions converge toward the exact result

- Example (#15, §16.2)

Evaluate the volume under the graph of the function $z=f(x,y)=\dfrac{y}{x^2+y^2}$ for the region: $\Omega=\{(x,y)\mid y=x, y=2x, x=2\}$

This region is triangular (see figure below



So:

$$V=\int_0^2\int_x^{2x}\frac{y}{x^2+y^2}dydx=\int_0^2\left[\int_x^{2x}y\left(x^2+y^2\right)^{-1}dy\right]dx=\int_0^2\left[\tfrac{1}{2}\ln\left(x^2+y^2\right)\Big|_x^{2x}\right]dx=\tfrac{1}{2}\int_0^2\left(\ln 5x^2-\ln 2x^2\right)dx$$
$$=\tfrac{1}{2}\int_0^2\ln\left(\frac{5x^2}{2x^2}\right)dx=\tfrac{1}{2}\ln\left(\tfrac{5}{2}\right)\int_0^2dx=\ln\left(\tfrac{5}{2}\right)$$

- *POLAR COORDINATES*

Certainly there is nothing special about evaluating a double integral in Cartesian coordinates. Any coordinate system in $\mathbf{R}^2$ can serve as a useful representation. Polar coordinates in particular ease the facility of computation if the region or the function is expressed in terms of segments of conic sections. For any continuous *f* and simply connected Ω :

$$\iint_{\Omega} f(r,\theta)\,r\,dr\,d\theta = \iint_{\Omega} f(r,\theta)\,r\,d\theta\,dr = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{N} f(r_i,\theta_i)\,r_i\,\Delta\theta_i\,\Delta r_i = \lim_{N\to\infty,\Omega\_fixed}\sum_{i=1}^{N} f(r_i,\theta_i)\,r_i\,\Delta\theta_i\,\Delta r_i$$

Where (according to Fubini’s thm)

$$\iint_{\Omega} f(r,\theta)\,r\,dr\,d\theta = \int_{\alpha}^{\beta}\int_{r_1=f(\theta)}^{r_2=g(\theta)} f(r,\theta)\,r\,dr\,d\theta = \iint_{\Omega} f(r,\theta)\,r\,d\theta\,dr = \int_{r_1}^{r_2}\int_{\theta_1=h[\theta]}^{\theta_2=l(\theta)} f(r,\theta)\,r\,d\theta\,dr$$

Moreover:

$$\iint_{\Omega} f(x,y)\,dx\,dy = \iint_{\Omega} f(x,y)\,dy\,dx = \iint_{\Omega} f(r,\theta)\,r\,dr\,d\theta = \int_{\alpha}^{\beta}\int_{r_1=f(\theta)}^{r_2=g(\theta)} f(r,\theta)\,r\,dr\,d\theta = \int_{r_1}^{r_2}\int_{\theta_1=h[\theta]}^{\theta_2=l(\theta)} f(r,\theta)\,r\,d\theta\,dr$$

(According to change of variables)

- Example: (# 11, §16.3)

  Find the area of the function: $r = 2\sin 3\theta$

  This is a 3 petaled leaf. Without loss of generality:

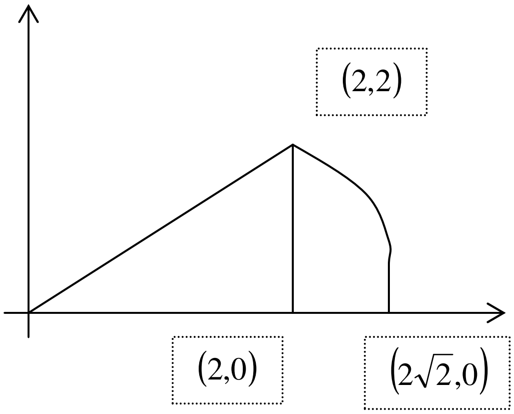
$$A = 3\int_{0}^{\pi/3}\int_{0}^{2\sin 3\theta} r\,dr\,d\theta = 3\int_{0}^{\pi/3}\left[\int_{0}^{2\sin 3\theta} r\,dr\right]d\theta = \tfrac{3}{2}\int_{0}^{\pi/3} r^2\Big|_{0}^{2\sin 3\theta}\,d\theta$$

$$= \tfrac{3}{2}\int_{0}^{\pi/3} 4\sin^2 3\theta\,d\theta = 6\int_{0}^{\pi/3}\sin^2 3\theta\,d\theta = 3\int_{0}^{\pi/3}(1-\cos 6\theta)\,d\theta = 3\theta\Big|_{0}^{\pi/3} - \tfrac{1}{2}\sin 6\theta\Big|_{0}^{\pi/3} = \pi$$

- Example (# 19, §16.3)

Evaluate: $\displaystyle\int_{0}^{2}\int_{0}^{x}\sqrt{x^2+y^2}\,dy\,dx + \int_{2}^{2\sqrt{2}}\int_{0}^{\sqrt{8-x^2}}\sqrt{x^2+y^2}\,dy\,dx$

The region (see below is specified as):



$$\therefore \int_0^2\int_0^x \sqrt{x^2+y^2}\,dydx + \int_2^{2\sqrt{2}}\int_0^{\sqrt{8-x^2}} \sqrt{x^2+y^2}\,dydx = \int_0^{\pi/4}\int_0^{2\sqrt{2}} r\cdot r\,drd\theta$$

$$= \int_0^{\pi/4}\int_0^{2\sqrt{2}} r^2\,drd\theta = \int_0^{\pi/4}\left[\tfrac{1}{3}r^3\Big|_0^{2\sqrt{2}}\right]d\theta = \frac{8^{3/2}}{3}\int_0^{\pi/4} d\theta = \tfrac{16}{3}\sqrt{2}\cdot\tfrac{\pi}{4} = \tfrac{4}{3}\pi\sqrt{2}$$

- <u>Example (# 29, §16.3)</u>

Find volume hemisphere $z=\sqrt{16-x^2-y^2}$ and inside cylinder $x^2+y^2-4x=0$

Completing square on cylinder: $x^2+y^2-4x=0 \Rightarrow (x-2)^2+(y-0)^2=2^2$ indicates in the x-y plane a circle centered at $x = 2$, $y = 0$, with radius 2. In polar coordinates: $x^2+y^2-4x=0 \Rightarrow r^2-4r\cos\theta=0 \Rightarrow r=4\cos\theta$

Without loss of generality, we can evaluate the result in the first octant and multiply the answer by (due to symmetry about x-y plane and symmetry about x-z planes)

$$Vol = 4\int_0^{\pi/2}\int_0^{4\cos\theta}\sqrt{16-r^2}\,r\,drd\theta = 4\int_0^{\pi/2}\left[\int_0^{4\cos\theta}\sqrt{16-r^2}\,r\,dr\right]d\theta = -2\int_0^{\pi/2}\left[\tfrac{2}{3}\left(16-r^2\right)^{3/2}\right]_0^{4\cos\theta}d\theta$$

$$= \tfrac{4}{3}\int_{\pi/2}^{0}\left[64\sin^3\theta-64\right]d\theta = \tfrac{256}{3}\int_{\pi/2}^{0}\left(1-\cos^2\theta\right)\sin\theta\,d\theta - \tfrac{256}{3}\int_{\pi/2}^{0}d\theta = \tfrac{256}{3}\left[-\cos\theta+\tfrac{1}{3}\cos^3\theta+-1\right]_{\pi/2}^{0}$$

$$= \tfrac{256}{3}\left[-1+\tfrac{1}{3}+\tfrac{\pi}{2}\right] = \tfrac{256}{3}\left(\tfrac{\pi}{2}-\tfrac{2}{3}\right) = \tfrac{256}{3}\left(\tfrac{3\pi-4}{6}\right) = \tfrac{128}{9}\left(3\pi-4\right)$$

Note: the book's answer is $\tfrac{64}{9}(3\pi-4)$ hence one may assume the volume is calculated above the x-y plane