MA262 **Notes for April 22**
Spring 2008 Kallfelz

- *PARTIAL INTEGRATION (THE INDEFINITE CASE)*

Now that you've seen the role that partial differentiation plays in cases of functions with vector domains which map to scalars (i.e. functions $f : \mathbf{R}^n \to \mathbf{R}$, where $n \geq 2$), the natural question is how to define anti-differentiation in such cases? Consider the following questions:

1. Suppose $\Phi_{xy}(x, y) = \frac{\partial^2}{\partial y \partial x}\Phi(x, y) = x \sin xy$. Find $\Phi(x, y)$.
2. Suppose $\vec{\nabla}\phi(x, y, z) = (zy + \sin y)\hat{\imath} + (x \cos y + xz - z^2 y)\hat{\jmath} + (-zy^2 + xy)\hat{k}$.
   Find $\phi(x, y, z)$.

In both cases, the anti-derivatives of such functions must be obtained. However, the procedure is via *indefinite partial integration.*

In the first case:

$$\Phi_x = \int \Phi_{xy}(x, y)dy = \int x \sin xy dy = x\int \sin xy dy = x\left(-\tfrac{1}{x}\cos xy\right) + f(x) = -\cos xy + f(x)$$
$$\therefore \Phi = \int \Phi_x(x, y)dx = \int (-\cos xy + f(x))dx = -\tfrac{1}{y}\sin xy + F(x) + g(y)$$

...where $f(x)$ and $g(x)$ are homogeneous functions of x and y respectively, and $F(x)$ is $f(x)$'s antiderivative. These are the analogues of the undetermined constants of integration in the ordinary case of indefinite integration.

Note how (just as in the case of partial differentiation) the other variable(s) (i.e. those not being the variable(s) of integration) were treated as though they were constants.

Unless more information is specified (in the form of boundary conditions and/or initial conditions) the aforementioned homogeneous functions remain unspecified. Nevertheless, one can always check the answer:

$$\frac{\partial}{\partial x}\Phi = \frac{\partial}{\partial x}\left(-\tfrac{1}{y}\sin xy + F(x) + g(y)\right) = -\cos xy + f(x)$$
$$\frac{\partial}{\partial y}\Phi_x = \frac{\partial^2}{\partial y \partial x}\Phi = \frac{\partial}{\partial y}(-\cos xy + f(x)) = x \sin xy$$

In the second case, more information is provided to give a definite answer. Since:

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{\imath} + \frac{\partial \phi}{\partial y}\hat{\jmath} + \frac{\partial \phi}{\partial z}\hat{k}$$

Then:

$$\frac{\partial}{\partial x}\phi = zy + \sin y \Rightarrow \phi(x,y,z) = \int (zy + \sin y)dx = xyz + x\sin y + f(y,z)$$
$$\frac{\partial}{\partial y}\phi = x\cos y + xz - z^2 y \Rightarrow \phi(x,y,z) = \int (x\cos y + xz - z^2 y)dy = x\sin y + xyz - \tfrac{1}{2}z^2y^2 + g(x,z)$$
$$\frac{\partial}{\partial z}\phi = -zy^2 + xy \Rightarrow \int (-zy^2 + xy)dz = -\tfrac{1}{2}z^2y^2 + xyz + h(x,y)$$

...where f , g, h are $\mathbf{R^2} \rightarrow \mathbf{R}$ functions (of ordered pair: $\langle y,z\rangle$, $\langle x,z\rangle$, $\langle x,y\rangle$ respectively). Based on the above three answers, f , g, h have "no choice" but to assume the following representations:

$f(y, z) = -\tfrac{1}{2}z^2y^2$, $g(x,z) = C$, $h(x, y) = x\sin y$

Hence: $$\phi(x,y,z) = xyz - \tfrac{1}{2}z^2y^2 + x\sin y + C ,$$

…which can be easily checked:

$$\frac{\partial}{\partial x}\phi = \frac{\partial}{\partial x}\left(xyz - \tfrac{1}{2}z^2y^2 + x\sin y + C\right) = yz + \sin y$$
$$\frac{\partial}{\partial y}\phi = \frac{\partial}{\partial y}\left(xyz - \tfrac{1}{2}z^2y^2 + x\sin y + C\right) = xz - z^2y + x\cos y$$
$$\frac{\partial}{\partial z}\phi = \frac{\partial}{\partial z}\left(xyz - \tfrac{1}{2}z^2y^2 + x\sin y + C\right) = xy - zy^2$$

Notice how in both cases, the other variables (i.e. those that weren't the variables of integration) were treated as "constants" in the integration process (in exact accord with the procedure of partial differentiation). Additionally, the "constants" of integration become functions solely involving the variables that weren't part of the integration. In the first case involving the $\mathbf{R^2} \rightarrow \mathbf{R}$ map, the "constants" were homogeneous functions, whereas in the second case involving the $\mathbf{R^3} \rightarrow \mathbf{R}$ map, the "constants" were $\mathbf{R^2} \rightarrow \mathbf{R}$ functions not involving the variable of integration.

- *PARTIAL DEFINITE INTEGRATION*
  The above procedure suggests the following in the case of definite integration:

$$\int_{g_1(y)}^{g_2(y)} \Phi_{xy}(x,y)dx = \Phi_y(x,y)\Big|_{g_1(y)}^{g_2(y)} = \Phi_y(g_2(y),y) - \Phi_y(g_1(y),y)$$
$$\int_{h_1(x)}^{h_2(x)} \Phi_{xy}(x,y)dy = \Phi_x(x,y)\Big|_{h_1(x)}^{h_2(x)} = \Phi_x(x,h_2(x)) - \Phi_x(x,h_1(x))$$

(when applying the Fundamental Theorem of Calculus:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

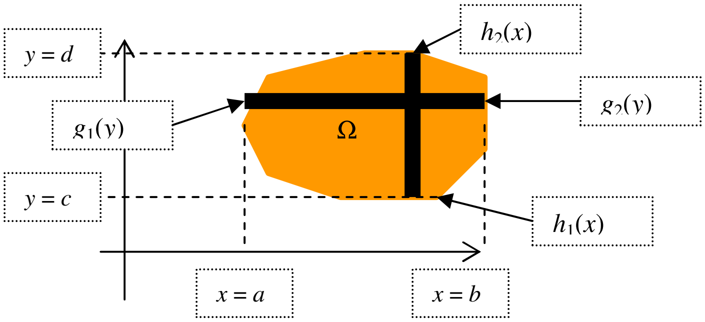
…where $F'(x) = f(x)$

In the above partial definite integrals, the 'constant' upper and lower limits of integration become *functions of the variables not involved in the integral*. As indicated on the right hand side, the expressions become homogeneous functions with respect to x and y. According to Fubini's Theorem (Thm 16.2, p. 942, text), as long as $\Phi_{xy} = \frac{\partial^2 \Phi}{\partial y \partial x}$ is continuous, on any simply connected region[1] Ω in $\mathbf{R}^2$**:**

$$\iint_\Omega \Phi_{xy}(xy)dxdy = \iint_\Omega \Phi_{xy}(x,y)dydx$$

I.e., as in the case of the order of differentiation for mixed second-order partial derivatives (for functions with continuous second-order partial derivatives) the order of integration is irrelevant.

Such a simply connected region Ω in $\mathbf{R}^2$ can always be represented either as bounded above and below by some functions $h_1(x)$, $h_2(x)$ or from the left and right by some functions $g_1(y)$, $g_2(y)$, as suggested by the figure below:



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[1] I.e., one with no holes or subdivided partitions. Stated more precisely, a simply connected region Ω in $\mathbf{R}^2$ is one in which**; i) (Ω is connected)** For *any* pair of points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ inside Ω, there exists a path $\vec{r}(t)$ (where the path is represented as a parametric curve in $\mathbf{R}^2$) such that $\vec{r}(t)$ connects $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ and $\vec{r}(t)$ lies entirely inside Ω. **ii) (Ω has no holes)** There does not exist any nonempty subregion $\Omega' \subset \Omega$ such that $\Omega \cap (\mathbf{R}^2 - \Omega')$ is nonempty, where: $\mathbf{R}^2 - \Omega' = \{<x, y> \mid <x, y> \notin \Omega'\}$.

…which suggests:

$$\iint_{\Omega} \Phi_{xy}(xy)\,dxdy = \int_{c}^{d}\int_{h_1(y)}^{h_2(y)} \Phi_{xy}\,dxdy = \int_{a}^{b}\int_{g_1(x)}^{g_2(x)} \Phi_{xy}\,dydx = \iint_{\Omega} \Phi_{xy}(x,y)\,dydx$$

…which characterizes the *double partial definite integral* for $\mathbf{R}^2 \to \mathbf{R}$ functions.

- **Note 1**: The above (single) partial integrals then become resolved via the double integral procedure in the following manner:

$$\int_{c}^{d}\int_{g_1(y)}^{g_2(y)} \Phi_{xy}(x,y)\,dxdy = \int_{c}^{d} \left(\Phi_{y}(x,y)\Big|_{g_1(y)}^{g_2(y)}\right)dy = \int_{c}^{d} \left(\Phi_{y}(g_2(y),y) - \Phi_{y}(g_1(y),y)\right)dy$$

$$\int_{a}^{b}\int_{h_1(x)}^{h_2(x)} \Phi_{xy}(x,y)\,dydx = \int_{a}^{b} \left(\Phi_{x}(x,y)\Big|_{h_1(x)}^{h_2(x)}\right)dx = \int_{a}^{b} \left(\Phi_{x}(x,h_2(x)) - \Phi_{x}(x,h_1(x))\right)dx$$

- **Note 2**: Fubini’s theorem suggests the following equivalence for continuous $\mathbf{R}^3 \to \mathbf{R}$ functions:

$$\int_{a}^{b}\int_{g_1(z)}^{g_2(z)}\int_{\varphi_1(y,z)}^{\varphi_2(y,z)} \Phi_{xyz}(x,y,z)\,dxdydz = \int_{a}^{b}\int_{f_1(z)}^{f_2(z)}\int_{\psi_1(x,z)}^{\psi_2(x,z)} \Phi_{xyz}(x,y,z)\,dydxdz = \int_{c}^{d}\int_{h_1(y)}^{h_2(y)}\int_{\phi_1(x,y)}^{\phi_2(x,y)} \Phi_{xyz}(x,y,z)\,dzdxdy$$

$$= \int_{c}^{d}\int_{k_1(y)}^{k_2(y)}\int_{\lambda_1(y,z)}^{\lambda_2(y,z)} \Phi_{xyz}(x,y,z)\,dxdzdy = \int_{e}^{g}\int_{l_1(x)}^{l_2(x)}\int_{\vartheta_1(x,z)}^{\vartheta_2(x,z)} \Phi_{xyz}(x,y,z)\,dydzdx = \int_{e}^{g}\int_{m_1(x)}^{m_2(x)}\int_{\gamma_1(x,y)}^{\gamma_2(x,y)} \Phi_{xyz}(x,y,z)\,dzdydx$$

- <u>Example (§16.1, #7)</u>

$$\int_{e^y}^{y} y\frac{\ln x}{x}dx = y\int_{e^y}^{y} \ln x\left(\frac{dx}{x}\right) = y\int_{\ln e^y = y}^{\ln y} udu = y\left\{\tfrac{1}{2}u^2\Big|_{y}^{\ln y}\right\} = \tfrac{1}{2}y\left((\ln y)^2 - y^2\right)$$

- <u>Example (§16.1, #9)</u>

$$\int_{0}^{x^3} ye^{-y/x}dy \Rightarrow u(y) = y, du = dy, dv(y) = e^{-y/x}, v(y) = -xe^{-y/x}$$

$$\Rightarrow \int_{0}^{x^3} ye^{-y/x}dy = -yxe^{-y/x}\Big|_{0}^{x^3} + x\int_{0}^{x^3} e^{-y/x}dx = \left[-yxe^{-y/x} - x^2e^{-y/x}\right]_{0}^{x^3}$$

$$= -x^4e^{-x^2} + 0 - x^2e^{-x^2} + x^2 = x^2\left(1 - e^{-x^2} - x^2e^{-x^2}\right)$$

- Example (§16.1, #15)

$$\int_0^1\int_0^{\sqrt{1-y^2}}(x+y)dxdy=\int_0^1\left[\int_0^{\sqrt{1-y^2}}(x+y)dx\right]dy=\int_0^1\left[\left(\tfrac{1}{2}x^2+yx\right)\Big|_0^{\sqrt{1-y^2}}\right]dy$$

$$=\int_0^1\left[\tfrac{1}{2}\left(1-y^2\right)+y\sqrt{1-y^2}\right]dy=\tfrac{1}{2}\int_0^1\left(1-y^2\right)dy+\int_0^1 y\sqrt{1-y^2}\,dy$$

$$=\tfrac{1}{2}\left(y-\tfrac{1}{3}y^3\right)\Big|_0^1+\int_{u(0)=1}^{u(1)=0}\left(-\tfrac{1}{2}du\right)u^{1/2}=\tfrac{1}{3}+\tfrac{1}{2}\int_0^1 u^{1/2}du=\tfrac{1}{3}+\tfrac{1}{2}\cdot\tfrac{2}{3}u^{3/2}\Big|_0^1=\tfrac{2}{3}$$

- Example (§16.1, #19)

$$\int_0^{\pi/2}\int_0^{\sin\theta}\theta r\,dr\,d\theta=\int_0^{\pi/2}\theta\left[\int_0^{\sin\theta}r\,dr\right]d\theta=\int_0^{\pi/2}\theta\left[\tfrac{1}{2}r^2\Big|_0^{\sin\theta}\right]d\theta=\tfrac{1}{2}\int_0^{\pi/2}\theta\left(\sin^2\theta-0\right)d\theta$$

$$=\tfrac{1}{2}\int_0^{\pi/2}\theta\sin^2\theta\,d\theta=\tfrac{1}{4}\int_0^{\pi/2}\theta(1-\cos 2\theta)d\theta=\tfrac{1}{4}\int_0^{\pi/2}\theta\,d\theta-\tfrac{1}{4}\int_0^{\pi/2}\theta\cos 2\theta\,d\theta$$

$$=\tfrac{1}{8}\theta^2\Big|_0^{\pi/2}-\tfrac{1}{4}\left[\tfrac{1}{2}\theta\sin 2\theta\Big|_0^{\pi/2}-\tfrac{1}{2}\int_0^{\pi/2}\sin 2\theta\,d\theta\right]=\tfrac{1}{32}\pi^2-\tfrac{1}{4}\left[0+\tfrac{1}{4}\cos 2\theta\Big|_0^{\pi/2}\right]$$

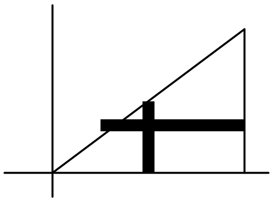
$$=\tfrac{1}{32}\pi^2-\tfrac{1}{16}\left(\cos\pi-\cos 0\right)=\tfrac{1}{32}\pi^2+\tfrac{1}{8}=\tfrac{1}{32}\left(\pi^2+4\right)$$

Note how the second integral was integrated by parts:
$u(\theta)=\theta\rightarrow du=d\theta, dv(\theta)=\cos 2\theta\rightarrow v(\theta)=\tfrac{1}{2}\sin 2\theta$

- Example (§16.1, #12) (modified)

Verify Fubini's Theorem for $\int_0^1\int_0^x\sqrt{1-x^2}\,dydx$

This region is depicted by region: $0 \le y \le x$, $0 \le x \le 1$, which (when interchanging variables of integration) becomes: $y \le x \le 1$, $0 \le y \le 1$ (see below)



$$\int_0^1\int_0^x \sqrt{1-x^2}\,dydx = \int_0^1\left[\int_0^x \sqrt{1-x^2}\,dy\right]dx = \int_0^1\sqrt{1-x^2}\left[\int_0^x dy\right]dx = \int_0^1 \sqrt{1-x^2}\left[y\Big|_0^x\right]dx$$
$$= \int_0^1 x\sqrt{1-x^2}\,dx = \int_{u(0)=1}^{u(1)=0}\left(-\tfrac{1}{2}du\right)u^{1/2} = -\tfrac{1}{2}\cdot\tfrac{2}{3}u^{3/2}\Big|_1^0 = \tfrac{1}{3}u^{3/2}\,|_0^1 = \tfrac{1}{3}$$

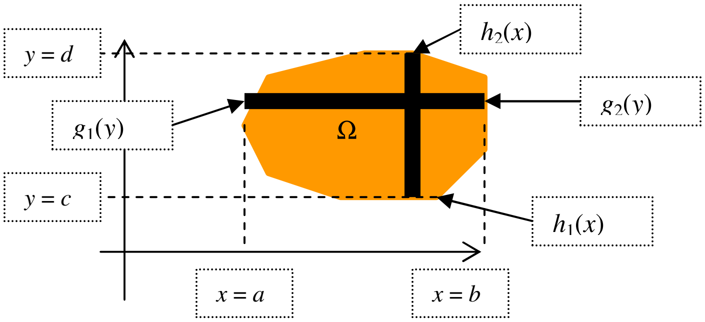
$$\int_0^1\int_y^1 \sqrt{1-x^2}\,dxdy = \int_0^1\left[\int_{\arcsin y}^{\arcsin 1=\pi/2}\sqrt{1-\sin^2\theta}\,\cos\theta d\theta\right]dy = \int_0^1\left[\int_{\arcsin y}^{\pi/2}\cos^2\theta d\theta\right]dy$$
$$= \int_0^1\left[\tfrac{1}{2}\int_{\arcsin y}^{\pi/2}(1+\cos 2\theta)d\theta\right]dy = \tfrac{1}{2}\int_0^1\left[\theta+\tfrac{1}{2}\sin 2\theta\Big|_{\arcsin y}^{\pi/2}\right]dy = \tfrac{1}{2}\int_0^1\left[\tfrac{\pi}{2}-\arcsin y+0-\sin(\arcsin y)\cos(\arcsin y)\right]dy$$
$$= \tfrac{1}{2}\int_0^1\left[\tfrac{\pi}{2}-\arcsin y-y\sqrt{1-y^2}\right]dy = \tfrac{1}{2}\left[\tfrac{\pi}{2}y-y\arcsin y-\sqrt{1-y^2}+\tfrac{1}{2}\cdot\tfrac{2}{3}(1-y^2)^{3/2}\right]_0^1$$
$$= \tfrac{1}{2}\left[(\tfrac{\pi}{2}-\arcsin 1-0)-(0-0-1+\tfrac{1}{3})\right] = \tfrac{1}{2}\left[(\tfrac{\pi}{2}-\tfrac{\pi}{2}-0)-(-\tfrac{2}{3})\right] = \tfrac{1}{2}\cdot\tfrac{2}{3} = \tfrac{1}{3}$$

- *SPECIAL CASE: AREA*

Consider the special case of the integrand = 1. This is a constant function, and obviously continuous. Hence according to Fubini's Theorem:

$$\iint_\Omega dxdy = \int_c^d\int_{h_1(y)}^{h_2(y)} dxdy = \int_a^b\int_{g_1(x)}^{g_2(x)} dydx = \iint_\Omega dydx = A$$

Which is just the area of the region:



You have seen how to compute areas in Calculus I [2], here the concept is cast into more general form, expressed as a double-integral

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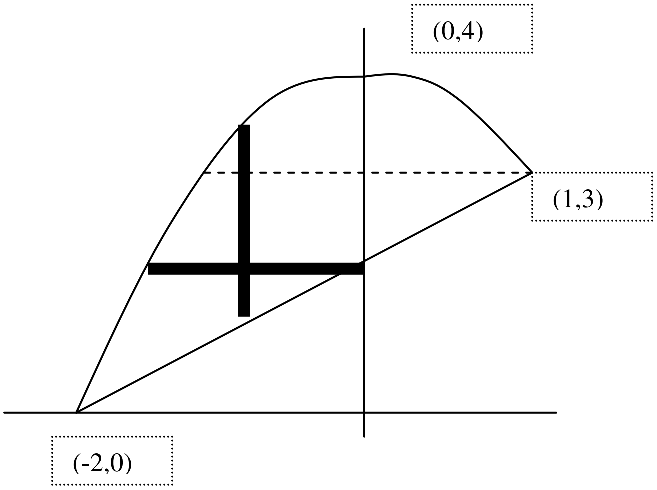
[2] For more details, see: http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Oct11notesb.pdf

- Example (§16.1, #41) (modified)

Verify Fubini's Thm. to calculate the area shared by curves $y = x + 2$, $y = 4 - x^2$

Finding their points of intersection: $x + 2 = 4 - x^2 \to x^2 + x - 2 = (x + 2)(x - 1) = 0$



$$\iint_{\Omega} dydx = \int_{-2}^{1}\int_{x+2}^{4-x^2} dydx = \int_{-2}^{1} y\Big|_{x+2}^{4-x^2} dx = \int_{-2}^{1}\left[4 - x^2 - x - 2\right]dx$$
$$= \int_{-2}^{1}\left(2 - x^2 - x\right)dx = \left(2x - \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right)\Big|_{-2}^{1} = \left(2 - \tfrac{1}{3} - \tfrac{1}{2}\right) - \left(-4 + \tfrac{8}{3} - 2\right) = 8 - \tfrac{5}{6} - \tfrac{8}{3}$$
$$= 8 - \tfrac{21}{6} = \tfrac{27}{6} = \tfrac{9}{2}$$

$$\iint_{\Omega} dxdy = \int_{c}^{d}\int_{h_1(y)}^{h_2(y)} dxdy = \int_{0}^{3}\int_{-\sqrt{4-y}}^{y-2} dxdy + \int_{3}^{4}\int_{-\sqrt{4-y}}^{\sqrt{4-y}} dxdy = \int_{0}^{3}\left[y - 2 + \sqrt{4-y}\right]dy + \int_{3}^{4} 2\sqrt{4-y}\,dy$$
$$= \left[\tfrac{1}{2}y^2 - 2y - \tfrac{2}{3}(4-y)^{3/2}\right]_0^3 - \tfrac{4}{3}(4-y)^{3/2}\Big|_3^4 = \left[\tfrac{9}{2} - 6 - \tfrac{2}{3} + \tfrac{16}{3}\right] - \tfrac{4}{3}(0-1) = \tfrac{9}{2} - 6 + \tfrac{18}{3} = \tfrac{9}{2}$$

- Example

  Evaluate: $\int_{0}^{\infty} e^{-x^2}\,dx$

$$\left\{\int_0^\infty e^{-x^2}dx\right\}^2=\int_0^\infty e^{-x^2}dx\int_0^\infty e^{-y^2}dy=\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}dxdy=\int_0^{2\pi}\int_0^\infty e^{-r^2}rdrd\theta$$

$$=\int_0^{2\pi}\left[-\tfrac{1}{2}e^{-r^2}\Big|_0^\infty\right]d\theta=-\tfrac{1}{2}\int_0^{2\pi}(\lim_{b\to\infty}e^{-b^2}-1)d\theta=\tfrac{1}{2}\int_0^{2\pi}d\theta=\pi$$

$$\therefore\int_0^\infty e^{-x^2}dx=\sqrt{\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-\left(x^2+y^2\right)}dxdy}=\sqrt{\pi}$$

- *GEOMETRIC INTERPRETATION OF THE DOUBLE (DEFINITE) INTEGRAL*

  In the case of the ordinary definite integral of f(x), expressed as a limit of a Riemann sum, the definite integral for a particular finite segment of the domain $[a, b]$ represents the *area* under the graph of f(x), whenever $f(x) > 0$ (and the negative value of area in the opposite case $f(x) \leq 0$). By the same token, for a simply connected finite region Ω of $\mathbf{R^2}$ **,** double definite integral for $z = g(x, y)$ represents the *volume* under the segment of the surface of the graph of $g(x, y)$ so long as $z > 0$, subtended by Ω (and the negative value of volume in the opposite case $z \leq 0$). Hence the double (and triple) integral obeys all the same properties of the Riemann integral as in the ordinary case.

- Example (# 33, §16.2)

  Find the volume in the first octant under the surface: $z(x, y) = x + y$ subtended by region: Ω: $x^2 + y^2 = 4$

$$Vol=\int_0^2\int_0^{\sqrt{4-x^2}}(x+y)dydx=\int_0^2\left[\int_0^{\sqrt{4-x^2}}(x+y)dy\right]dx=\int_0^2\left[xy+\tfrac{1}{2}y^2\Big|_0^{\sqrt{4-x^2}}\right]dx=\int_0^2\left[x\sqrt{4-x^2}+\tfrac{1}{2}\left(4-x^2\right)\right]dx$$

$$=\int_{u(0)=4}^{u(2)=0}\left(-\tfrac{1}{2}du\right)u^{1/2}+\left[2x-\tfrac{1}{6}x^3\right]\Big|_0^2=\tfrac{1}{3}u^{3/2}\Big|_0^4+4-\tfrac{4}{3}=\tfrac{8}{3}-\tfrac{4}{3}+4=\tfrac{16}{3}$$

As in the previous example, this integral is simpler to evaluate when switching to polar coordinates (since Ω is a circular segment):

$$Vol=\int_0^2\int_0^{\sqrt{4-x^2}}(x+y)dydx=\int_0^{\pi/2}\int_0^2(r\cos\theta+r\sin\theta)rdrd\theta=\int_0^{\pi/2}\left[(\cos\theta+\sin\theta)\int_0^2 r^2dr\right]$$

$$=\int_0^{\pi/2}(\cos\theta+\sin\theta)\cdot\tfrac{1}{3}r^3\Big|_0^2=\int_0^{\pi/2}\tfrac{8}{3}(\sin\theta+\cos\theta)d\theta=\tfrac{8}{3}\left[-\cos\theta+\sin\theta\right]_0^{\pi/2}=\tfrac{8}{3}\left(0+1-(-1+0)\right)=\tfrac{16}{3}$$