- *BASICS OF MULTIVARIATE (DOMAIN) FUNCTIONS (Cont.)*

As **April 15 notes**[1] summarized in cursory detail, functions with multivariate (vector-valued) domains and scalar valued ranges share most of the same properties you've encountered in the garden-variety $f : R \to R$ case, i.e. mappings from scalars to scalars. Nevertheless, there are some subtle differences (the introduction of the concept of partial derivative being one of them). Here I continue with some of the basics (not involving calculus).

- **Finding domain/range**

Recall[2] that for any mapping or function: $f : X \to Y$ (where *X* and *Y* are sets):

1. The **Domain** of *f* (i.e. **Dom*f*) =** $\{x \in X \mid f(x)\_is\_well\_defined\}$**.** Basically, in Calculus I and II, "well defined" means that for *y* = *f*(*x*): $-\infty < y < \infty$ (i.e. *y* = *f*(*x*) is a real number and finite).
2. The **Range** of *f* (i.e. **Range*f*) =** $\{y \in Y \mid y = f(x) \,\&\, x \in Domf\}$**,** i.e. the set of all points that *f* maps from its domain. Sometimes the Range of *f* is denoted as the **Image** of *f*.

- **Note1:** It goes without saying that: $Domf \subseteq X, Rangef \subseteq Y$. That is to say, *f*'s domain and range ***need not*** include all of *X* and *Y* (i.e. the domain and range of *f* ca be proper subsets of *X* and *Y*, respectively). For example, in the case of the $f : R \to R$ mapping, where: *f*(*x*) = arcsin*x*:

$$Domf = \{x \mid f(x)\_is\_well\_defined\} = \{x \mid -1 \le x \le 1\} = [-1,1]$$
$$Rangef = \{y \mid y = f(x), x \in Domf\} = \{y \mid y = \arcsin x, x \in [-1,1]\} = \{y \mid -\tfrac{\pi}{2} \le y \le \tfrac{\pi}{2}\} = [\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

Obviously, these sets are very restricted subsets of the real numbers *R* = (- ∞ , ∞)

- Example (# 11, § 15.1)

Find the domain/range of the mapping: $f : R^2 \to R : f(x, y) = \sqrt{4 - x^2 - y^2}$

$$Domf = \{\vec{r} \mid f(\vec{r})\_is\_well\_defined\} = \{\vec{r} = \langle x, y\rangle \mid 4 - x^2 - y^2 \ge 0\}$$
$$= \{(x, y) \mid x^2 + y^2 \le 2^2\} = \overline{D}_2(0)$$

($\overline{D}_2(0)$: A closed[3] disk centered at the origin, of radius 2

[1] http://www.glue.umd.edu/~wkallfel/MA261-2/Apr15.pdf
[2] Either from Calculus I or from some other math course

$$Rangef = \{z \mid z = f(\vec{r}), \vec{r} \in Domf\} = \left\{z \mid z = \sqrt{4 - x^2 - y^2}, \langle x, y \rangle \in \overline{D}_2(0)\right\} = \{z \mid 0 \le z \le 2\} = [0,2]$$

- Example (# 13, § 15.1)

Find the domain/range of the mapping: $f : R^2 \to R : \ f(x, y) = \arcsin(x + y)$

$$Domf = \{\vec{r} \mid f(\vec{r})\_is\_well\_defined\} = \{\vec{r} = \langle x, y \rangle \mid -1 \le x + y \le 1\}$$
$$= \{(x, y) \mid -1 - x \le y \le 1 - x\}$$

An infinite closed[4] strip, bounded below by the line: $y = -1 - x$, and bounded above by the line: $y = 1 - x$

$$Rangef = \{z \mid y = f(\vec{r}), \vec{r} \in Domf\} = \{z \mid z = \arcsin(x + y), \langle x, y \rangle \in Domf\} = \{z \mid -\tfrac{\pi}{2} \le z \le \tfrac{\pi}{2}\} = [\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

- Example

Find the domain/range of the mapping: $f : R^3 \to R : \ f(x, y, z) = \ln(x + y + 2z)$

$$Domf = \{\vec{r} \mid f(\vec{r})\_is\_well\_defined\} = \{\vec{r} = \langle x, y, z \rangle \mid 0 < x + y + 2z\}$$

I.e., the open[5] region in $\mathbf{R^3}$ lying above the plane: $x + y + 2z = 0$

$$Rangef = \{z \mid y = f(\vec{r}), \vec{r} \in Domf\} = \{w \mid w = \ln(x + y + 2z), \langle x, y, z \rangle \in Domf\} = \{w \mid -\infty < w < \infty\} = (\infty, \infty)$$

As mentioned in p. 7, **April 15 notes**[6] , the graphs of mappings of $f : R^n \to R$ mappings (for $n > 2$) cannot be directly represented. Finding however the **level surfaces** (in the case of $f : R^3 \to R$ mapping) as well as the **level curves** (in the case of $f : R^2 \to R$ mapping[7]) can provide useful information, insofar as specifying regions in the domain which all map to a constant point c in the range.

- Example (# 49, § 15.1)

Find the level curves for the mapping $f : R^2 \to R : \quad f(x, y) = \arctan\left(\frac{y}{x}\right)$

$c = 0, \pm\tfrac{\pi}{6}, \pm\tfrac{\pi}{3}, \pm\tfrac{5}{6}\pi$

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[3] In other words, including its boundary
[4] I.e., including its boundary
[5] Not including boundary
[6] http://www.glue.umd.edu/~wkallfel/MA261-2/Apr15.pdf
[7] Whose graph *can* be directly represented

$c = 0 = \arctan(y/x) \Rightarrow \tan 0 = \frac{y}{x} \Rightarrow \{\langle x, y\rangle \mid y = 0, x \neq 0\}$ (i.e., the x-axis minus the origin)

$c = \pm\frac{\pi}{6} = \arctan(y/x) \Rightarrow \tan\pm\frac{\pi}{6} = \pm\frac{\sqrt{3}}{3} = \frac{y}{x} \Rightarrow \{\langle x, y\rangle \mid y = \pm\frac{\sqrt{3}}{3}x\}$ (i.e., the set of lines passing through the origin with slope $\pm\frac{\sqrt{3}}{3}x$

$c = \pm\frac{\pi}{3} = \arctan(y/x) \Rightarrow \tan\pm\frac{\pi}{3} = \pm\sqrt{3} = \frac{y}{x} \Rightarrow \{\langle x, y\rangle \mid y = \pm\sqrt{3}x\}$ (i.e., the set of lines passing through the origin with slope $\pm\sqrt{3}x$

$c = \pm\frac{5\pi}{6} = \arctan(y/x) \Rightarrow \tan\pm\frac{5\pi}{6} = \mp\frac{\sqrt{3}}{3} = \frac{y}{x} \Rightarrow \{\langle x, y\rangle \mid y = \pm\frac{\sqrt{3}}{3}x\}$ (i.e., the set of lines passing through the origin with slope $\mp\frac{\sqrt{3}}{3}x$ (equivalent to the first case)

- Example (# 49, § 15.1)

Find the level surfaces for the mapping $f: R^3 \to R$: $f(x, y, z) = 4(x^2 + y^2) - z^2$ for the case: $w = f(x,y,z) = c$

In the case $c = 0$: $0 = 4(x^2 + y^2) - z^2$: i.) In the $x - y$ plane ($z = 0$) $0 = 4(x^2 + y^2) \Rightarrow x = y = 0$ (the origin). Furthermore note that for any value of z = c > 0, $c = 4(x^2 + y^2) \Rightarrow x^2 + y^2 = \left(\frac{c}{2}\right)^2$ (a circle of radius $c/2$) ii.) In the $x - z$ plane ($y = 0$) $0 = 4x^2 - z^2 \Rightarrow z = \pm 2x$ (a pair of lines intersecting the origin with slope ± 2). iii.) In the $y - z$ plane ($x = 0$) $0 = 4y^2 - z^2 \Rightarrow z = \pm 2y$ (a pair of lines intersecting the origin with slope ± 2). All three projections suggest a pair of *cones* intersecting the origin (of slope 2 in the positive octant in the x-z and y-z planes).

- *LIMITS OF MULTIVARIATE FUNCTIONS*

Recall from Calculus I[8]:

> $\lim_{x \to x_0} f(x) = L$ if and only if for all $\varepsilon > 0$ such that $0 < |f(x) - L| < \varepsilon$, there exists a $\delta(\varepsilon)$ such that $0 < |x - x_0| < \delta$ and the ε-dependency of δ is such that as: $\varepsilon \to 0$ then $\delta(\varepsilon) \to 0$.

[8] For more information, see pp. 9-10 of **Aug 30** posting (Calc I) http://www.glue.umd.edu/%7Ewkallfel/MA261-2/MA261Aug30notes.pdf

In the case of $f : R^n \to R$ mappings (for n = 2 and higher), the above definition is basically the same. The subtle modifications of course regard the vector nature of the points in the domain:

$\lim_{\vec{r}\to\vec{r}_0} f(\vec{r}) = L$ if and only if for all $\varepsilon > 0$ such that $0 < |f(\vec{r}) - L| < \varepsilon$, there exists a $\delta(\varepsilon)$ such that $0 < \|\vec{r} - \vec{r}_0\| < \delta$ and the ε-dependency of δ is such that as: $\varepsilon \to 0$ then $\delta(\varepsilon) \to 0$.

Now, to spell this out further, note that for instance in the case of a $f : R^2 \to R$ mapping, $\lim_{\vec{r}\to\vec{r}_0} f(\vec{r}) = L$ means: $\lim_{\langle x,y\rangle\to\langle x_0,y_0\rangle} f(x,y) = L$ and $0 < \|\vec{r} - \vec{r}_0\| < \delta$ means:

$$\{(x,y) \mid 0 < \|\langle x,y\rangle - \langle x_0,y_0\rangle\| < \delta\} = \{(x,y) \mid \|\langle x - x_0, y - y_0\rangle\| < \delta\} = \left\{(x,y) \mid 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta\right\}$$
$$= \left\{(x,y) \mid 0 < (x-x_0)^2 + (y-y_0)^2 < \delta^2\right\} = D_\delta(0)$$

…which is an open punctured[9] disk centered at the origin of radius δ.

In the case of a $f : R^3 \to R$ mapping, $\lim_{\vec{r}\to\vec{r}_0} f(\vec{r}) = L$ means: $\lim_{\langle x,y,z\rangle\to\langle x_0,y_0,z_0\rangle} f(x,y,z) = L$ and $0 < \|\vec{r} - \vec{r}_0\| < \delta$ means:

$$\{(x,y) \mid 0 < \|\langle x,y,z\rangle - \langle x_0,y_0,z_0\rangle\| < \delta\} = \left\{(x,y) \mid 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} < \delta\right\}$$
$$= \left\{(x,y) \mid 0 < (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 < \delta^2\right\} = D_\delta(0)$$

…which is an open punctured[10] disk centered at the origin of radius δ.

Recall also from Calculus I that for a function to be **continuous at x_0,** then: $\lim_{x\to x_0} f(x) = f(x_0)$, which entails three necessary conditions: i.) $\lim_{x\to x_0} f(x) = L$ exists, ii.) $f(x_0)$ exists, iii.) $L = f(x_0)$

The same notions carry over in the case of a $f : R^n \to R$ mapping, i.e. f is **continuous** at $\vec{r}_0$ then: $\lim_{\vec{r}\to\vec{r}_0} f(\vec{r}) = f(\vec{r}_0)$, which entails three necessary conditions: i.) $\lim_{\vec{r}\to\vec{r}_0} f(\vec{r}) = L$ exists, ii.) $f(\vec{r}_0)$ exists, iii.) $L = f(\vec{r}_0)$, where the limit (in i.)) is defined via the above description of limits of $f : R^n \to R$ mappings

[9] I.e., not including the boundary nor its center point
[10] I.e., not including the boundary nor its center point

- *MIXED PARTIALS AND CONTINUITY*

Higher-order partials can be mixed or homogeneous, i.e. they can involve the same variable(s) or different ones in different orders. Consider for example some general $f : R^3 \to R$ mapping. There are a total of *nine* cases for its second-order partial derivatives:

$$f_{xx} = \frac{\partial^2 f}{\partial^2 x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \qquad f_{xz} = \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)$$
$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \qquad f_{yy} = \frac{\partial^2 f}{\partial^2 y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \qquad f_{yz} = \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)$$
$$f_{zx} = \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) \qquad f_{zy} = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) \qquad f_{zz} = \frac{\partial^2 f}{\partial^2 z} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)$$

- Note2: Observe the ordering in the subscripts in the Newtonian notation: reading from left to right, the first variable indicates which partial derivative is evaluated first. In other words, reading left to right indicates the parenthetical ordering of the operations.

However, whenever the function is continuous, as well as all its partial derivatives up to and including all its second-order partials, then its second-order mixed partials are equal (Thm 15.3, p. 868), so:

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right),$$
$$f_{zx} = \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) = f_{xz} = \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right),$$
$$f_{zy} = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = f_{yz} = \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)$$

Whenever the condition of continuity up to and including all its second order partials holds. So in the case of a $f : R^3 \to R$, this reduces the number of independent cases of second order partials from nine to six (as well as in the $f : R^2 \to R$ case, the number of independent cases get reduced from four to three.

- <u>Example (# 49, § 15.3</u>)

Show that $f : R^3 \to R$ mapping $w = f(x, y, z) = e^{-x}\sin(yz)$ satisfies the above.

Note that this function is the product and composition of differentiable functions (to any arbitrary order) so perhaps it's not too surprising that the equality of mixed partials holds. Nevertheless:

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} e^{-x}\sin(yz)\right) = \frac{\partial}{\partial x}\left(e^{-x} z\cos(yz)\right) = -e^{-x} z\cos(yz)$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} e^{-x}\sin(yz)\right) = \frac{\partial}{\partial y}\left(-e^{-x}\sin(yz)\right) = -e^{-x} z\cos(yz)$$

$$f_{yz} = \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y} e^{-x}\sin(yz)\right) = \frac{\partial}{\partial z}\left(e^{-x} z\cos(yz)\right) = e^{-x}\left(\cos(yz) - yz\sin(yz)\right)$$

$$f_{zy} = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial z} e^{-x}\sin(yz)\right) = \frac{\partial}{\partial y}\left(e^{-x} y\cos(yz)\right) = e^{-x}\left(\cos(yz) - yz\sin(yz)\right)$$

$$f_{xz} = \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x} e^{-x}\sin(yz)\right) = \frac{\partial}{\partial z}\left(-e^{-x}\sin(yz)\right) = -e^{-x} y\cos(yz)$$

$$f_{zx} = \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial z} e^{-x}\sin(yz)\right) = \frac{\partial}{\partial x}\left(e^{-x} y\cos(yz)\right) = -e^{-x} y\cos(yz)$$

Hence it is evident that the mixed partials are all equal.

- *TOTAL DERIVATIVE/DIFFERENTIAL/CHAIN RULE*

Aside from partially differentiating a multivariate function, which means fixing all its independent variables while varying just one, one can also define a *total* derivative:

In the case of a $f : R^2 \to R$ mapping, for instance:

$$\frac{d}{dx} f(x,y) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \Rightarrow df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = f_x dx + f_y dy$$

..which is an example of the *chain rule* in action. For when differentiating the function by *x*, one considers the case in which *f* varies with respect to *x* (i.e. the first term: $\frac{\partial f}{\partial x}$), but there is the second possibility which we add[11] which involves the *y*-dependency on *x*, note how the chain rule is adopted: First take the partial derivative of *f* with respect to *y* and then multiply by the derivative of *y* with respect to *x*.

In the case of a $f : R^3 \to R$ mapping the same reasoning of course holds, and hence one may write its total differential as:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = f_x dx + f_y dy + f_z dz$$

From this reasoning, one can consider more general cases like *x*, *y*, *z* themselves depending (in a perhaps multivariate sense) on other parameters *s*, *t*, *u*…

For instance, consider $f : R^3 \to R$ mapping in which: $x(s, t)$, $y(s, t)$, $z(s,t)$; i.e. *x*, *y*, *z* themselves are multivariate functions, depending on two parameters *s*,*t*. Then in accordance with the above procedure for calculating $\frac{d}{dx} f$:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} = f_x x_s + f_y y_s + f_z z_s$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} = f_x x_t + f_y y_t + f_z z_t$$

…Which is an instance of the *general chain rule*, applied to the case of partial derivatives.

- <u>Example :</u> Suppose $f : R^3 \to R$, where: $f(x, y, z) = \ln x \tan\left(\frac{y}{z}\right)$

  a.) Find its total differential

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = f_x dx + f_y dy + f_z dz$$
$$= \frac{1}{x}\tan\left(\frac{y}{z}\right)dx + \ln x \frac{1}{z}\sec^2\left(\frac{y}{z}\right)dy - \ln x \frac{y}{z^2}\sec^2\left(\frac{y}{z}\right)dz$$

[11] This follows the general **addition principle:** to describe a total number of possible outcomes, which are all mutually disjunct, one simply adds up all the outcomes. In other words, (exclusive) "OR" means "+" in mathematics. For example, suppose I told you that for dessert you get two choices for fruit OR three choices for pudding (but you cannot have both pudding and fruit). So your total number of choices then are 2 + 3 = 5 options.

b.) Suppose $x(s,t)=s^2t,\, y(s,t)=\sin(st),\, z(s,t)=s+t$. Use the information in a.) above to find: $\dfrac{\partial f}{\partial s}$

$$\frac{\partial f}{\partial s}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial s}=f_x x_s+f_y y_s+f_z z_s$$
$$=\frac{1}{x(s,t)}\tan\left(\frac{y(s,t)}{z(s,t)}\right)\frac{\partial x}{\partial s}+\frac{\ln x(s,t)}{z(s,t)}\sec^2\left(\frac{y(s,t)}{z(s,t)}\right)\frac{\partial y}{\partial s}-\frac{\ln x(s,t)y(s,t)}{z^2(s,t)}\sec^2\left(\frac{y(s,t)}{z(s,t)}\right)\frac{\partial z}{\partial s}$$

Now: $\dfrac{\partial x}{\partial s}=2st,\dfrac{\partial y}{\partial s}=t\cos(st),\dfrac{\partial z}{\partial s}=1$. Inserting and specifying explicitly the *s*, *t* dependency of *x,y,z* :

$$\frac{\partial f}{\partial s}=\frac{1}{x(s,t)}\tan\left(\frac{y(s,t)}{x(s,t)}\right)\frac{\partial x}{\partial s}+\frac{\ln x(s,t)}{z(s,t)}\sec^2\left(\frac{y(s,t)}{z(s,t)}\right)\frac{\partial y}{\partial s}-\frac{\ln x(s,t)y(s,t)}{z^2(s,t)}\sec^2\left(\frac{y(s,t)}{z(s,t)}\right)\frac{\partial z}{\partial s}$$
$$=\frac{\tan(\sin(st)/(s+t))}{s^2t}2st+\frac{\ln(s^2t)}{s+t}\sec^2\left(\frac{\sin(st)}{(s+t)}\right)t\cos(st)-\frac{\ln(s^2t)\sin(st)}{(s+t)^2}\sec^2\left(\frac{\sin(st)}{s+t}\right)$$
$$=\frac{2}{s}\tan\left(\frac{\sin(st)}{s+t}\right)+\frac{(2\ln s+\ln t)t\cos(st)}{s+t}\sec^2\left(\frac{\sin(st)}{s+t}\right)-\frac{(2\ln s+\ln t)\sin(st)}{(s+t)^2}\sec^2\left(\frac{\sin(st)}{s+t}\right)$$
$$=\frac{2}{s}\tan\left(\frac{\sin(st)}{s+t}\right)+\frac{(2\ln s+\ln t)}{s+t}\sec^2\left(\frac{\sin(st)}{s+t}\right)\left[t\cos(st)-\frac{\sin(st)}{s+t}\right]$$

- *DIRECTIONAL DERIVATIVE AND GRADIENT*

Note that the total differential expression derived above via the chain rule lends itself naturally to a ‘dot product’ characterization:

In the case of a $f:R^2\to R$ mapping

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=f_x dx+f_y dy=\langle f_x,f_y\rangle\cdot\langle dx,dy\rangle=\left(\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}\right)\cdot\left(dx\hat{i}+dy\hat{j}\right)$$

In the case of a $f:R^3\to R$ mapping

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz=f_x dx+f_y dy+f_z dz$$
$$=\langle f_x,f_y,f_z\rangle\cdot\langle dx,dy,dz\rangle=\left(\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}\right)\cdot\left(dx\hat{i}+dy\hat{j}+dz\hat{k}\right)$$

The first vector is known as the *gradient* vector of *f*, and denoted: $\vec{\nabla} f$. The second vector in the dot product we recognize as the infinitesimal change in position: $d\vec{r}$. Hence the exact differential may be stated succinctly by a dot product: $df = \vec{\nabla} f \cdot d\vec{r}$

- **Note 3:** So far, we've covered cases of total differentiation for $f : R \to R^n$ ($n > 1$) mappings as well as for $f : R^n \to R$ ($n > 1$) mappings, i.e. scalar → vector mappings, as well as vector → scalar mappings. In the latter case, it has been shown that the total derivative is defined in terms of an n – component *vector* (the gradient vector). What about the most general case, i.e. vector → vector mappings, i.e. of the form: $f : R^m \to R^n$ ($n > 1$, $m > 1$) ? It turns out that in such a case, the total derivative *Df* is defined via an $n \times m$ *matrix*, otherwise known as the Jacobi matrix:

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix},$$

where the subscripts refer to a general component of an *m*-dimensional vector in the domain as well as some *n*-dimensional vector in the range. For shorthand, the Jacobi matrix is often depicted as: $Df = \lfloor \partial_i f_j \rfloor$, where: $\partial_i \equiv \frac{\partial}{\partial x_i}$, and it's understood that: $1 \le i \le m, 1 \le j \le n$. There are other ways of representing the above matrix. For instance, if we write the range values specifically as an *n*-dimensional column vector: $\vec{f}(\vec{r}) = \begin{pmatrix} f_1(\vec{r}) \\ f_2(\vec{r}) \\ \vdots \\ f_n(\vec{r}) \end{pmatrix}$, where $\vec{r}$ is an *m*-dimensional *row* vector in *f*'s domain: $\vec{r} = \langle x_1, x_2, \cdots, x_m \rangle$, then note that the above Jacobi matrix can be cast in *row*-vector form:

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix} = \left\langle \frac{\partial}{\partial x_1}\vec{f}, \frac{\partial}{\partial x_2}\vec{f}, \cdots, \frac{\partial}{\partial x_n}\vec{f} \right\rangle$$

On the other hand, recognizing that each row is a *gradient* vector, one can re-express the Jacobi matrix in column-vector form:

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix} = \begin{bmatrix} \vec{\nabla} f_1 \\ \vec{\nabla} f_2 \\ \vdots \\ \vec{\nabla} f_n \end{bmatrix}$$

For example, consider the case of the *electric field*. It is defined as a 3D vector in every point in 3D space, therefore, the electric field is an $\mathbf{R^3} \rightarrow \mathbf{R^3}$ mapping i.e.: $E : R^3 \rightarrow R^3$, where: $\vec{E}(\vec{r}) = E_1(x,y,z)\hat{i} + E_2(x,y,z)\hat{j} + E_3(x,y,z)\hat{k}$ (Denote the *x,y,z* components of the field by the subscripts 1,2,3 so as not to confuse this with the Newtonian notation for partial differentiation.)

Hence the electric field's Jacobi matrix is expressed as:

$$D\vec{E} = \begin{bmatrix} \frac{\partial E_1}{\partial x} & \frac{\partial E_1}{\partial y} & \frac{\partial E_1}{\partial z} \\ \frac{\partial E_2}{\partial x} & \frac{\partial E_2}{\partial y} & \frac{\partial E_2}{\partial z} \\ \frac{\partial E_3}{\partial x} & \frac{\partial E_3}{\partial y} & \frac{\partial E_3}{\partial z} \end{bmatrix} = \left\langle \frac{\partial}{\partial x}\vec{E}, \frac{\partial}{\partial y}\vec{E}, \frac{\partial}{\partial z}\vec{E} \right\rangle = \begin{bmatrix} \vec{\nabla} E_1 \\ \vec{\nabla} E_2 \\ \vec{\nabla} E_3 \end{bmatrix}$$

The gradient vector has some interesting properties. Among other things, *it always points in a direction of maximum change*. To see this, consider the (special) case of a *directional* derivative; i.e. a measure of the rate of change of the function *along some direction* denoted (of course) by the unit vector $\hat{u}$, defined as:[12] $D_{\hat{u}} f = \vec{\nabla} f \cdot \hat{u}$

Now, for instance, in the case of a $f : R^2 \rightarrow R$ mapping, any unit vector in R^2 (as shown in **<u>April 7th notes</u>**[13]) can be represented as $\hat{u} = \cos\theta\hat{i} + \sin\theta\hat{j}$, where θ is the polar angle depicting the direction of interest (measured counterclockwise from the $+x$ axis.)

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[12] In this regard, it's apparent that the *partial* derivatives are a special case of a directional derivative, measuring the rate of change of the function in the *x*, *y*, *z* directions respectively. This is explicitly shown (for example) in the case of the *x*-direction: $D_{\hat{i}} f = \vec{\nabla} f \cdot \hat{i} = \left(f_x\hat{i} + f_y\hat{j} + f_z\hat{k}\right)\cdot \hat{i} = f_x = \frac{\partial f}{\partial x}$

[13] http://www.glue.umd.edu/%7Ewkallfel/MA261-2/Apr7.pdf

So in this case: $D_{\hat{u}} f = \vec{\nabla} f \cdot \hat{u} = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} \right) \cdot \left(\cos\theta\hat{i} + \sin\theta\hat{j} \right) = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$

Since the maximum possible values of cosine and sine =1, this quantity doesn't exceed the values of the sum of the values of the components of the gradient vector.

Moreover, recall for any two vectors: $\vec{u}, \vec{v} : \left| \vec{u} \cdot \vec{v} \right| \le \left\| \vec{u} \right\| \left\| \vec{v} \right\|$.

Hence: $\left| D_{\hat{u}} f \right| = \left| \vec{\nabla} f \cdot \hat{u} \right| \le \left\| \vec{\nabla} f \right\| \left\| \hat{u} \right\| = \left\| \vec{\nabla} f \right\|$

(I.e the value of the directional derivative is bounded above by the magnitude of the gradient)

- Example (§15.6, #19) Find the directional derivative for the $f : R^3 \to R$ mapping: $h(x, y, z) = \ln(x + y + z)$ along a path from P(1,0,0) to Q(4,3,1):

1. A unit vector in the direction of PQ is: $\hat{u} = \frac{PQ}{\left\| PQ \right\|} = \frac{\langle (4-1), (3-0), (1-0) \rangle}{\sqrt{3^2 + 3^2 + 1^2}} = \frac{1}{\sqrt{19}} \langle 3,3,1 \rangle = \frac{\sqrt{19}}{19} \left(3\hat{i} + 3\hat{j} + \hat{k} \right)$

2. The gradient of the function is:

$$\vec{\nabla} h = h_x \hat{i} + h_y \hat{j} + h_z \hat{k} = \frac{1}{x + y + z} \left(\hat{i} + \hat{j} + \hat{k} \right)$$

3. Hence: $D_{\hat{u}} h = \vec{\nabla} h \cdot \hat{u} = \frac{\sqrt{19}}{19} \left(\frac{3 + 3 + 1}{x + y + z} \right) = \frac{\frac{7}{19}\sqrt{19}}{x + y + z}$

4. However note that x, y, z lie on a line segment from P to Q. Using the same unit vector as the line segment's direction vector, the equation of the line can be represented as $\vec{r}(t) = \vec{r}_0 + t\hat{u} = \langle 1,0,0 \rangle + \frac{\sqrt{19}}{19} t \langle 3,3,1 \rangle$. Hence x,y,z take on the following parametric representations:

   $$x(t) = 1 + \frac{3\sqrt{19}}{19} t,\ y(t) = \frac{3\sqrt{19}}{19} t,\ z(t) = \frac{\sqrt{19}}{19} t$$

   Hence, inserting:

   $$D_{\hat{u}} h = \vec{\nabla} h \cdot \hat{u} = \frac{\frac{7}{19}\sqrt{19}}{x + y + z} = \frac{\frac{7}{19}\sqrt{19}}{1 + \frac{7}{19}\sqrt{19}t} = \frac{1}{\frac{19}{7\sqrt{19}} + t} = \frac{1}{\frac{\sqrt{19}}{7} + t}$$

5. Note how the above answer can be checked directly via the chain rule (provided the same parameterization is adopted):

$$\frac{dh}{dt}=\frac{\partial h}{\partial x}\frac{dx}{dx}+\frac{\partial h}{\partial y}\frac{dy}{dt}+\frac{\partial h}{\partial z}\frac{dz}{dt}=\frac{1}{x+y+z}\left[\tfrac{3}{19}\sqrt{19}+\tfrac{3}{19}\sqrt{19}+\frac{\sqrt{19}}{19}\right]=\frac{\tfrac{7}{19}\sqrt{19}}{x(t)+y(t)+z(t)}$$

$$=\frac{\tfrac{7}{19}\sqrt{19}}{1+\tfrac{7}{19}\sqrt{19}t}=\frac{1}{\frac{\sqrt{19}}{7}+t}$$

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