- *HINTS (ASSIGNMENT 3)*

Exercise II.b)

$$x = a\cos^3\theta,\ y = a\sin^3\theta \Rightarrow ds = \sqrt{(x'(\theta))^2 + (y'(\theta))^2} = \sqrt{(-3a\cos^2\theta\sin\theta)^2 + (3a\sin^2\theta\cos\theta)^2}$$
$$= 3a\sqrt{\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta} = 3a\sqrt{\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)} = 3a\sqrt{\cos^2\theta\sin^2\theta}$$
$$= 3a\cos\theta\sin\theta$$

Exercise II.b)

$$x = a\cos\theta,\ y = b\sin\theta \Rightarrow ds = \sqrt{(x'(\theta))^2 + (y'(\theta))^2} = \sqrt{(-a\sin\theta)^2 + (b\cos\theta)^2}$$
$$= \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$$
$$(a.): dS = 2\pi yd = 2\pi(b\sin\theta)\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} = 2\pi b\sin\theta\sqrt{a^2(1-\cos^2\theta) + b^2\cos^2\theta}$$
$$= 2\pi b\sin\theta\sqrt{a^2 + (b^2 - a^2)\cos^2\theta} = 2\pi ab\sin\theta\sqrt{1 + \left(\frac{b^2 - a^2}{a^2}\right)\cos^2\theta} = 2\pi ab\sin\theta\sqrt{1 - e^2\cos^2\theta}$$
$$(e^2 = \frac{a^2 - b^2}{a^2})$$

Let $u = e\cos\theta$, then: $du = -e\sin\theta d\theta$. So:

$$S = 2\pi ab\int_0^{\pi}\sin\theta\sqrt{1 - e^2\cos^2\theta}\,d\theta = -\frac{2\pi ab}{e}\int_{u(0)=e}^{u(\pi)=-e} du\sqrt{1-u^2} = \tfrac{2\pi ab}{e}\int_{-e}^{e} du\sqrt{1-u^2}$$

At this point, you can do another (simple) trig substitution, or just look up the antiderivative in the table (Formula 37, **Appendix C**)

$$(b.): dS = 2\pi xd = 2\pi(b\cos\theta)\sqrt{a^2\sin^2\theta + b^2\cos^2\theta} = 2\pi b\cos\theta\sqrt{b^2(1-\sin^2\theta) + a^2\sin^2\theta}$$
$$= 2\pi b\cos\theta\sqrt{b^2 + (a^2 - b^2)\sin^2\theta} = 2\pi b^2\cos\theta\sqrt{1 + \left(\frac{a^2 - b^2}{b^2}\right)\sin^2\theta} = 2\pi b^2\cos\theta\sqrt{1 - \tfrac{b^2}{a^2}e^2\sin^2\theta}$$
$$(e^2 = \frac{a^2 - b^2}{a^2})$$

Let $u = {}^{b}/_{a}e\sin\theta$, then: $du = {}^{b}/_{a}e\cos\theta,\ d\theta$. So:

$$S = 2\pi b^2\int_0^{\pi}\cos\theta\sqrt{1 - \tfrac{b^2}{a^2}e^2\sin^2\theta}\,d\theta = \frac{2\pi ab}{e}\int_{u(0)=0}^{u(\pi)=eb/a} du\sqrt{1-u^2} = \tfrac{2\pi ab}{e}\int_0^{eb/a} du\sqrt{1-u^2}$$

As in case a.), you can do another (simple) trig substitution, or just look up the antiderivative in the table (Formula 37, **Appendix C**)

Exercise III.a)

$$r = 2\sin 3\theta = 2\sin(2\theta+\theta) = 2[\sin 2\theta\cos\theta + \cos 2\theta \sin\theta]$$
$$= 2[2\sin\theta\cos\theta\cos\theta + (\cos^2\theta - \sin^2\theta)\sin\theta] = 2[2\sin\theta\cos^2\theta + \cos^2\theta\sin\theta - \sin^3\theta]$$
$$= 2\sin\theta[2\cos^2\theta + \cos^2\theta - \sin^2\theta] = 2\sin\theta[3\cos^2\theta - \sin^2\theta]$$

Then insert the transformations: $x = r\cos\theta \Rightarrow \cos\theta = \dfrac{x}{r}$, $y = r\sin\theta \Rightarrow \sin\theta = \dfrac{y}{r}$

..carry all your r –terms over the left hand side. Don't forget the final step: $r = \sqrt{x^2+y^2}$

Exercise III.b)

Insert $x = r\cos\theta$, $y = r\sin\theta$ into the equation: $y^2 - 8x - 16 = 0$. Then use the quadratic equation to isolate r as a function of θ.

Exercise III.d)

$$x(\theta) = r(\theta)\cos\theta = e^{a\theta}\cos\theta \to x'(\theta) = ae^{a\theta}\cos\theta - e^{a\theta}\sin\theta$$
$$y(\theta) = r(\theta)\sin\theta = e^{a\theta}\sin\theta \to y'(\theta) = ae^{a\theta}\sin\theta + e^{a\theta}\cos\theta$$

$$\therefore ds = \sqrt{(x')^2 + (y')^2}$$
$$= \sqrt{a^2e^{2a\theta}\cos^2\theta - 2ae^{2a\theta}\cos\theta\sin\theta + e^{2a\theta}\sin^2\theta + a^2e^{2a\theta}\sin^2\theta + 2ae^{a\theta}\cos\theta\sin\theta + e^{2a\theta}\sin^2\theta}$$
$$= e^{a\theta}\sqrt{a^2(\cos^2\theta + \sin^2\theta) + (\sin^2\theta + \cos^2\theta)} = e^{a\theta}\sqrt{a^2+1}$$
$$\therefore S = 2\pi\int_0^{\pi/2} r(\theta)\cos\theta ds = 2\pi\int_0^{\pi/2} e^{a\theta}\cos\theta e^{a\theta}\sqrt{a^2+1}d\theta = 2\pi\sqrt{a^2+1}\int_0^{\pi/2} e^{2a\theta}\cos\theta d\theta$$

(which can be integrated by parts or looked up in **Appendix C**

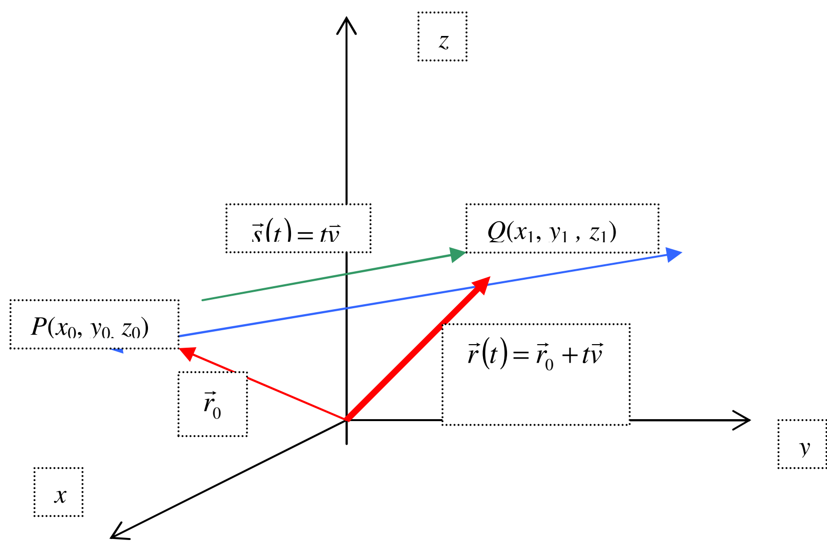
- *LINES AND PLANES IN* $\boldsymbol{R^3}$

  A **position vector,** denoted $\vec{r}(t)$ is anchored at the origin, and "points" to any point $P(x, y, z)$ in $\mathbf{R}^3$ (it "locates" the point as P is at the tip of the arrow). Hence by definition: $\vec{r}(t) = OP = \langle (x-0), (y-0), (z-0) \rangle = \langle x, y, z \rangle = x\hat{i} + y\hat{j} + z\hat{k}$.

  Just as in the case of $\mathbf{R}^2$**,** so in $\mathbf{R}^3$ **:** It takes two points to determine a line. Recall that in $\mathbf{R}^2$**,** given those two points $P(x_0, y_0)$, $Q(x_1, y_1)$ one can always determine the slope: $m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_1 - y_0}{x_1 - x_0}$. Similarly, given two points in $\mathbf{R}^3$**:** $P(x_0, y_0, z_0)$,

$Q(x_1,\ y_1,\ z_1)$, the analogy to the "slope" parameter in $\mathbf{R}^2$ the line's *direction* vector[1] $\vec{v} = PQ = \langle (x_1 - x_0), (y_1 - y_0), (z_1 - z_0) \rangle \equiv \langle a, b, c \rangle = a\hat{i} + b\hat{j} + c\hat{k}$. To ensure that the direction vector can "shrink" or "stretch" (like a rubber band that infinitely elastic) so as to roam up and down the line in space is ensured by multiplying the constant vector $\vec{v}$ with a real valued parameter t, i.e. form new (variable) vector: $\vec{s}(t) = \vec{v}t$. This is analogous to 'moving' up and down a line of slope m in $\mathbf{R}^2$ through the action of forming product: mx.

Hence, to specify the location of *any* point on the line in $\mathbf{R}^3$**,** simply form the vector sum: $\vec{r}(t) = \vec{r}_0 + \vec{s}(t) = \vec{r}_0 + \vec{v}t$, where: $\vec{r}_0$ is the (constant) position vector locating the first point $P(x_0,\ y_0,\ z_0)$ on the line. Note that it makes no difference which point we choose, i.e. we could have formed just as well: $\vec{r}(t) = \vec{r}_1 + \vec{s}(t) = \vec{r}_1 + \vec{v}t$, where: $\vec{r}_1$ is the (constant) position vector locating the second point $Q(x_1,\ y_1,\ z_1)$, on the line. Note the analogy of the form of this vector equation $\vec{r}(t) = \vec{r}_0 + \vec{s}(t) = \vec{r}_0 + \vec{v}t$ with the case of : $y = b + mx$ in $\mathbf{R}^2$. See figure below:



In coordinate form, then:

$$\vec{r}(t) = \vec{r}_0 + \vec{s}(t) = \vec{r}_0 + \vec{v}t = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle = \langle (x_0 + at), (y_0 + bt), (z_0 + ct) \rangle$$

$$= (x_0 + at)\hat{i} + (y_0 + bt)\hat{j} + (z_0 + ct)\hat{k}$$

[1] Note the order doesn't matter, i.e. one could have specified $\vec{v} = QP = \langle (x_0 - x_1), (y_0 - y_1), (z_0 - z_1) \rangle$ just as well. This is because when multiplying the direction vector by parameter t, t can take on *any* real value (negative or positive).

In component form: $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$, which when de-paramaterizing (by isolating *t*) one obtains the *symmetric equations of the line* in $\mathbf{R}^3$ **:**

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

- <u>Example:</u> Find the equation of the line passing through: *P*(1,-2,2), *Q*(2,0,-2)

  $$\vec{v} = PQ = \langle (2-1), (0--2), (-2-2) \rangle = \hat{i} + 2\hat{j} - 4\hat{k}$$
  $$\therefore \vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle 1,-2,2 \rangle + t\langle 1,2,-4 \rangle = (1+t)\hat{i} + (-2+2t)\hat{j} + (2-4t)\hat{k}$$

  Or in symmetric form: $t = x - 1 = \dfrac{y+2}{2} = \dfrac{2-z}{4}$

  Note that one could have chosen *Q* instead of *P* for the point, which would produce a different parameterization of the *same* line[2]:

  $$\therefore \vec{r}(t) = \vec{r}_1 + t\vec{v} = \langle 2,0,-2 \rangle + t\langle 1,2,-4 \rangle = (2+t)\hat{i} + (2t)\hat{j} + (-2-4t)\hat{k}$$

  Or in symmetric form: $t = x - 2 = \dfrac{y}{2} = \dfrac{-2-z}{4}$

  These two parameterizations yield the same line: to check this note that both *P* and *Q* are included in both parameterizations: In the first parameterization, when $t = 0$: $\vec{r}(0) = \vec{r}_0 = \langle 1,-2,2 \rangle$, i.e. locating *P*. In the second paramterization this occurs when $t = -1$: $\therefore \vec{r}(t) = \vec{r}_1 + -\vec{v} = (2-1)\hat{i} - (2)\hat{j} + (-2+4)\hat{k} = \langle 1,-2,2 \rangle$. Similar results hold for the case of *Q*

- *PLANES*

  Just as it takes two points to determine a line, it takes *three* (non-colinear) points $P(x_0, y_0, z_0)$, $Q(x_1, y_1, z_1)$, $R(x_2, y_2, z_2)$ to determine a plane. As shown in class (April 15th), one selects *any* point from the three in the plane to draw two (constant) vectors $\vec{a}, \vec{b}$ embedded in the plane and anchored at the same point. Suppose one chooses (arbitrarily) *P* as the anchor. Then: $\vec{a} = PQ, \vec{b} = PR$. One then forms a constant *normal* $\vec{n}$ (a vector perpendicular to the plane) via the cross product: $\vec{n} = \vec{a} \times \vec{b}$. Then form a *variable* vector $\vec{r}$ embedded in the plane anchored at P : $\vec{r} = XP$, where $X(x, y, z)$ is a point whose *x*, *y*, *z* values can vary.

[2] Recall from chapter 12: that *infinitely* many different parameterizations can characterize the same curve in space.

The conditions for $X(x, y, z)$ to remain in the plane is that (just like vectors $\vec{a}, \vec{b}$) $\vec{r} = XP$ must *always* be perpendicular to $\vec{n} = \vec{a} \times \vec{b}$, i.e.: $\vec{n} \cdot \vec{r} = 0$. The above condition will generate an equation: $Ax + By + Cz + D = 0$, where A, B, C are the x,y,z components of $\vec{n} = \vec{a} \times \vec{b}$

- Example: §14.3, # 23

  Find the equation of the plane containing $P(0,0,0)$, $Q(1,2,3)$, $R(-2,3,3)$

  Choose P (arbitratily) to anchor $\vec{a}, \vec{b}$: $\vec{a} = PQ = \langle 1-0, 2-0, 3-0 \rangle = \hat{i} + 2\hat{j} + 3\hat{k}$
  $\vec{b} = PR = \langle -2-0, 3-0, 3-0 \rangle = -2\hat{i} + 3\hat{j} + 3\hat{k}$.

$$\text{So: } \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -2 & 3 & 3 \end{vmatrix} = -3\hat{i} - 9\hat{j} + 7\hat{k}$$

  So, for any vector $\vec{r} = XP = \langle x-0, y-0, z-0 \rangle = x\hat{i} + y\hat{j} + z\hat{k}$, for it to remain in the plane, then $\vec{n} \cdot \vec{r} = 0 = \left(-3\hat{i} - 9\hat{j} + 7\hat{k}\right) \cdot \left(x\hat{i} + y\hat{j} + z\hat{k}\right) = -3x - 9y + 7z = 0$

When two planes intersect, one can determine the angle of intersection as well as the equation of their line of intersection. Reading off the coefficients of the two equations of the planes, one can construct their respective normal vectors: $\vec{n}_1 = A_1\hat{i} + B_1\hat{j} + C_1\hat{k}$, $\vec{n}_2 = A_2\hat{i} + B_2\hat{j} + C_2\hat{k}$. Then their angle is determined via the formula (recall previous handout): $\theta = \arccos\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \|\vec{n}_2\|}\right)$. To find their line of intersection amounts to eliminating (at least) one of the variables of the two equations (via substitution/elimination) and construction of a paramateric representation for the equation consisting of the other two variables. This is most easily done via taking the cross-product of $\vec{n}_1 = A_1\hat{i} + B_1\hat{j} + C_1\hat{k}$ and $\vec{n}_2 = A_2\hat{i} + B_2\hat{j} + C_2\hat{k}$ (this gives the direction vector of the line of intersection) and then obtaining finding a point on the plane to construct the equation of the line

- Example: Find the angle of intersections and the equation of the line of intersection between planes:

$$x + 2y + 3z = 0 \qquad -3x + 4y + z = 0$$

  So: $\vec{n}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$ $\qquad$ $\vec{n}_2 = -3\hat{i} + 4\hat{j} + \hat{k}$

Hence:

$$\theta = \arccos\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\|\|\vec{n}_2\|}\right) = \arccos\left(\frac{-3+8+3}{\sqrt{1+4+9}\sqrt{9+16+1}}\right) = \arccos\left(\frac{8}{\sqrt{26\cdot 14}}\right) \approx 1.14rad = 65.2^o$$

To find a point on the plane, since there are only two equations (and three unknowns) set one of the unknowns (choose one arbitrarily) = 0 and solve for the other two. Choosing x = 0: $\begin{matrix} 2y+3z=0 \\ 4y+z=0 \end{matrix} \Rightarrow y = 0 = z$, so the plane passes through the origin (another way of readily seeing this fact is that notice that in the above two equations, $D_1 = D_2 = 0$). Hence the direction vector of the line of intersection is:

$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -3 & 4 & 1 \end{vmatrix} = -10\hat{i} - 10\hat{j} + 10\hat{k}$$

So the parametric vector equation of the line of intersection is:

$\vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle 0,0,0\rangle + t\langle -10,-10,10\rangle = -10t\hat{i} - 10t\hat{j} + 10t\hat{k}$. Without loss of generality, the 10 (common factor) can be absorbed into the parameter t to yield a simpler equation of line of intersection: $\vec{r}(t) = -t\hat{i} - t\hat{j} + t\hat{k} = t\langle -1,-1,1\rangle$

- *MULTIVARIATE FUNCTIONS AND PARTIAL DERIVATIVES*

In the notes of the previous week, we have seen function of a single variable domain and a multivariate (vector) range. The simplest example of course are functions: $f : R \to R^2$, i.e. that map a real-valued scalar into a 2D vector. As shown, they're represented a vector:

$$\vec{f}(t) = x(t)\hat{i} + y(t)\hat{j} = \langle x(t), y(t)\rangle$$

where $x(t)$ and $y(t)$ are *functions* of t. Of course one can extend into $\mathbf{R}^3$ as well:

$$f : R \to R^3 \text{, where: } \vec{f}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} = \langle x(t), y(t), z(t)\rangle$$

Now, as long as the *domain* is 1D (a one dimensional scalar), as shown previously taking the derivative of such a function doesn't bring about any conceptual change: One simply differentiates the *components* of such a function:

$$D_t\vec{f}(t) = x'(t)\hat{i} + y'(t)\hat{j} = \langle x'(t), y'(t)\rangle \qquad \text{(in the } f : R \to R^2 \text{ case)}$$

Or: $D_t\vec{f}(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k} = \langle x'(t), y'(t), z'(t)\rangle$ in the $f: R \to R^3$ case.

Now, the question becomes: What happens in the *opposite* cases, i.e. for cases of functions of the type[3]: $f: R^2 \to R$, $f: R^3 \to R$? These kinds of functions have *multivariate* (vector-valued) domains but *scalar* valued range. Here are few simple examples:

1. The *length* of a vector (either 2D or 3D) is an example of a $f: R^2 \to R$ or $f: R^3 \to R$ mapping, respectively, with the following rule:

   In the $f: R^2 \to R$ case, $f(x, y) = \|\langle x, y\rangle\| = \sqrt{x^2 + y^2}$
   In the $f: R^3 \to R$ case, $f(x, y, z) = \|\langle x, y, z\rangle\| = \sqrt{x^2 + y^2 + z^2}$

   In other words, the action of *f* is to take a 2 (or 3) component vector and map it to a non-negative real number.

2. The *volume* of a cylinder can be thought of as a $f: R^2 \to R$ mapping: $f(r, h) = \pi r^2 h$, as *f* is taking a 2D vector (ordered pair) describing the cylinder's radius and height and mapping it to a non-negative real number.

3. The volume of a box, on the other hand, can be thought of as a $f: R^3 \to R$ mapping: $f(x, y, z) = xyz$, as *f* is taking a 3D vector (ordered triple) describing the box's length, width, height and mapping it to a non-negative real number.

The above three examples are meant to motivate your intuitions concerning these kinds of scalar -valued (in range) multivariate (i.e. vector, in domain) functions. Note, as discussed in 15.1, just as the graph of a $f: R \to R$ function ($y = f(x)$) produces a curve in $\mathbf{R^2}$, so the graph of a $f: R^2 \to R$ function ($z = f(x, y)$) produces a *surface* in $\mathbf{R^3}$. Furthermore, the graph of a $f: R^3 \to R$ function ($w = f(x, y, z)$) produces a *hypersurface* in $\mathbf{R^4}$. We're not equipped (of course) to "visualize" the graph of a $f: R^3 \to R$ function in $\mathbf{R^4}$ (it's a 4-dimensional space) so *direct representation*[4] of graphs of

[3] Of course more general kinds of cases of functions one may study (in Calculus and higher-level math) are of the variety: $f: R^n \to R^m$ (where *n* and *m* are positive integer). Such functions map *n*-dimensional vectors (in domain) to *m*-dimensional vectors (in range).

[4] Which is not to say one cannot *indirectly* get a sense of what's going on in the $f: R^3 \to R$ case: By holding various range values *fixed*, i.e. $f(x, y, z) = w_1, w_2, w_3, \ldots$ one generates the family of *level surfaces*

multivariate functions is extremely limited, namely only in the $f : R^2 \to R$ case.

The conceptual problem occurs when one seeks to define a *derivative* in these cases. The question is immediately raised: Derivative with respect to *which* independent variable? (For now there are several in the domain—at least two).

The above question is answered by way of the notion of the *partial* derivative. Recall from Calculus I the definition of the derivative of a $f : R \to R$ function: $y = f(x)$

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The *partial derivative* is defined with respect to the same limiting procedure (assuming *f* is differentiable)

In the $f : R^2 \to R$, $z = f(x, y)$ case:

$$\frac{\partial z}{\partial x} = f_x = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} \qquad \frac{\partial z}{\partial y} = f_y = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

In the $f : R^3 \to R$, $w = f(x, y, z)$ case:

$$\frac{\partial w}{\partial x} = f_x = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x, y, z)}{\Delta x}, \frac{\partial w}{\partial y} = f_y = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y, z) - f(x, y, z)}{\Delta y}$$

$$\frac{\partial w}{\partial y} = f_z = \lim_{\Delta z \to 0} \frac{f(x, y, z+\Delta z) - f(x, y, z)}{\Delta z}$$

Note the new notation: the "$\frac{\partial}{\partial}$" is the Leibnitzian notation, whereas the subscript is the Newtonian.

Conceptually, the partial derivative investigates the infinitesimal variation of the function with respect to a particular variable, *while holding all the other variables fixed* (i.e. treated as though they are constants)

(as discussed in 14.1, test) in the *domain* of *f*. The domain is of course $\mathbf{R}^3$, i.e. 3D space. But these level surfaces *are not the graph of the function*! They're just the regions in the domain of *f* in which its range values take on constant values. For instances in the case of the length of a vector in 3D, $f(x, y, z) = \|\langle x, y, z\rangle\| = \sqrt{x^2 + y^2 + z^2}$,the level surfaces of such a function a re family of spheres in $\mathbf{R}^3$

You may ask: Is there such a thing as a more general derivative (or total) derivative, analogous to $\frac{dy}{dx} = f'(x)$ in the multivariate case? In other words, are partial derivates the end of the story? Certainly not. The general notion of total derivative that applies in these cases is found by taking the following general limit procedure:

$$\lim_{\Delta\vec{r}\to\vec{0}} \frac{f(\vec{r}+\Delta\vec{r}) - f(\vec{r})}{\|\Delta\vec{r}\|}$$

As it turns out, a topic usually covered in broader detail in Calculus III, the above limiting procedure defines a *vector*, known as the *gradient* of the function, denoted $\vec{\nabla} f$:

In the $f : R^2 \to R$, $z = f(x, y)$ case: $$\vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}$$

In the $f : R^3 \to R$, $w = f(x, y, z)$ case: $$\vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$$

Note how the gradient vectors however are *defined* via the partial derivatives.

As mentioned, calculating partial derivatives is straightforward enough: hold all the *other* variables constant, which means *treat them as constants*:

- Example §14.3, # 19

$$z = f(x, y) = e^y \sin xy$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(e^y \sin xy\right) = e^y y \cos xy$$
$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(e^y \sin xy\right) = e^y \sin xy + e^y x \cos xy = e^y\left(\sin xy + x\cos xy\right)$$

- Example §14.3, # 21

$$f(x, y) = \int_x^y \left(t^2 - 1\right)dt = \left(\tfrac{1}{3}t^3 - t\right)\Big|_x^y = \left(\tfrac{1}{3}y^3 - y\right) - \left(\tfrac{1}{3}x^3 - x\right) = \tfrac{1}{3}\left(y^3 - x^3\right) - (y - x)$$

$$\frac{\partial}{\partial x} f = \frac{\partial}{\partial x}\left[\tfrac{1}{3}\left(y^3 - x^3\right) - (y - x)\right] = -x^2 + 1 \qquad \frac{\partial}{\partial y} f = \frac{\partial}{\partial y}\left[\tfrac{1}{3}\left(y^3 - x^3\right) - (y - x)\right] = y^2 - 1$$

- <u>Example §14.3, # 32</u>

$$f(x, y, z) = 3x^2 y - 5xyz + 10yz^2$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(3x^2 y - 5xyz + 10yz^2\right) = 6xy - 5yz$$
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(3x^2 y - 5xyz + 10yz^2\right) = 3x^2 - 5xz + 10z^2$$
$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(3x^2 y - 5xyz + 10yz^2\right) = -5xy + 20yz$$

- *NEXT CLASS: MIXED PARTIAL DERIVATIVES AND HIGHER-ORDER PARTIALS*