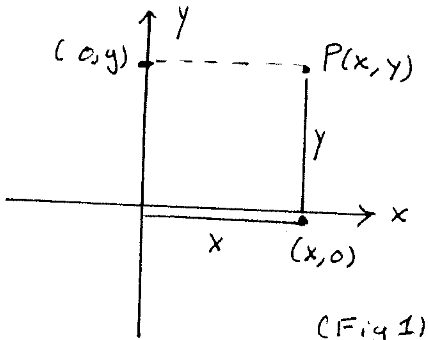
MA 262
SPRING 2008 KALLFELZ

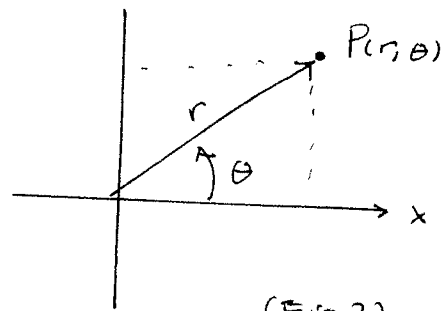
CALCULUS II : NOTES FOR APRIL 1ST

CARTESIAN
("BOX" COORDINATES)



(Fig. 1)

POLAR
("CIRCULAR" COORDINATES)



(Fig. 2)

SYMMETRY:

Recall (for Cartesian coordinates)

$f(x) = f(-x) \Rightarrow$ Symmetry about y-axis

$f(x) = -f(-x) \Rightarrow$ Symmetry about origin (i.e. about line $y=-x$)

- Convention: $\pm\theta$: COUNTER-CLOCKWISE (VS. CLOCKWISE (−) ROTATION ABOUT POLAR AXIS : $[0,\infty)$ " $\{x \mid 0 \le x < \infty$
- nonuniqueness : For every $P(r,\theta) = P(r, \theta + 2n\pi)$ $= P(-r, \theta + (2n+1)$

In the case of Polar coordinates, θ is (by convention) the independent variable, and r is the dependent variable. So all polar graphs are expressed as: $r = f(\theta)$ (f: some function).

SYMMETRY

- $r = f(\theta) = f(-\theta)$ or $r = -f(\pi - \theta) \Rightarrow$ Symmetric about x-axis.
- $r = f(\theta) = f(\pi - \theta)$ or $r = -f(-\theta) \Rightarrow$ Symmetric about y-axis
- $r = f(\theta) = f(\theta + \pi)$ or $r = -f(\theta) \Rightarrow$ Symmetric about origin.

TRANSFORMATION

AS Fig. 2 SUGGESTS, $x(\theta) = r\cos\theta \quad y(\theta) = r\sin\theta \Rightarrow \tan\theta = \frac{y}{x}$

$$\Downarrow$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

$$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta)$$
$$= r^2 \Rightarrow r = \pm\sqrt{x^2+y^2}$$

- POLAR → CARTESIAN : $\begin{cases} x(\theta) = r\cos\theta \\ y(\theta) = r\sin\theta \end{cases}$ $(r,\theta) \longrightarrow (x(\theta), y(\theta))$

- CARTESIAN → POLAR $\begin{cases} r(x,y) = \sqrt{x^2+y^2} \\ \theta(x,y) = \arctan(y/x) \end{cases}$ $(x,y) \longrightarrow (r(x,y), \theta(x,y))$

* Note : $-\infty < r < \infty$ and $-\infty < \theta < \infty$ (where θ is in radians).

However, for any θ: $-\frac{\pi}{2} \le \arctan\left(\frac{y}{x}\right) = \theta \le \frac{\pi}{2}$, so adjustments should be made for specifying completely all possible values of θ.

- Example 1 Find all θ such that $P_1(1,1)$, $P_2(-1,1)$, $P_3(1,-1)$, $P_4(-1,-1)$

$$\theta = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{\pm 1}{\pm 1}\right) = \arctan(\pm 1) = \pm \frac{\pi}{4}$$

However, the above values apply only in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

For all θ such that $\arctan\theta = \pm 1$ we have: $\frac{\pi}{4} + n\pi$
$-\frac{\pi}{4} + n\pi$

Combining the two cases $\Rightarrow \{\theta \mid \theta = \frac{\pi}{4} + n\pi\} \cup \{\theta \mid \theta = -\frac{\pi}{4} + n\pi\}$

$$\Rightarrow \{\theta \mid \theta = \frac{\pi}{4} + n\frac{\pi}{2} = \frac{\pi}{4}(1+2n)\}$$

So for all $\theta_n = \frac{\pi}{4}(2n+1) = (2n+1)\pi/4$, $\tan\theta_n = \pm 1$.

RECALL CONIC SECTIONS (SEE NOTES FOR MARCH 20)

A) Parabola (centered at origin) $y^2 = 4px$ (a) or $x^2 = 4py$ (b)

In polar form: $(r\sin\theta)^2 = 4p(r\cos\theta) \Rightarrow r^2\sin^2\theta = 4pr\cos\theta$

$$\Rightarrow r\sin^2\theta = 4p\cos\theta \Rightarrow r(\theta) = \frac{4p\cos\theta}{\sin^2\theta} = \frac{4p\cos\theta}{1-\cos^2\theta}$$

or $r(\theta) = \frac{4p\sin\theta}{1-\sin^2\theta}$ for case (b)

B) CIRCLE (CENTERED AT ORIGIN) $x^2 + y^2 = R^2 \Rightarrow r^2(\cos^2\theta + \sin^2\theta) = R^2$

$$\Rightarrow \boxed{r = R = \text{const}}$$

C) ELLIPSE (CENTERED AT ORIGIN) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

In polar form: $\frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1 \Rightarrow r^2\left(\cos^2\theta + \frac{a^2}{b^2}\sin^2\theta\right) = a^2$

$$\Rightarrow r^2\left[1 - \sin^2\theta + \frac{a^2}{b^2}\sin^2\theta\right] = a^2 \Rightarrow r^2\left[1 + \left(\frac{a^2}{b^2} - 1\right)\sin^2\theta\right] = a^2$$

$$\Rightarrow r^2\left[1 + \frac{a^2-b^2}{b^2}\sin^2\theta\right] = a^2 \Rightarrow r^2\left[1 + \frac{c^2}{a^2}\sin^2\theta\right] = a^2$$

$$\Rightarrow r = \pm \frac{a}{\sqrt{1+\frac{c^2}{a^2}\sin^2\theta}} \quad \text{where } \frac{c}{a} = e \text{ (eccentricity)} \Rightarrow r = \pm \frac{a}{\sqrt{1+e^2\sin^2\theta}}$$

(note: $e = \frac{c}{a} = \frac{\sqrt{a^2-b^2}}{a} < 1$

D) HYPERBOLA (CENTERED AT ORIGIN)

$$\overset{(a)}{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1} \quad \text{or} \quad \overset{(b)}{\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1}$$

$$r = \pm \overset{(a)}{\frac{a}{\sqrt{1+e^2\sin^2\theta}}} \quad \text{where } e = \frac{c}{a} > 1 \quad c = \sqrt{a^2+b^2}$$

or

$$r = \pm \frac{a}{\sqrt{1-e^2\cos^2\theta}} \quad \text{(case (b))}$$

Certainly, for the general cases, i.e. rotated and centered not at origin, the algebra can get messier when $(x, y) \longleftrightarrow (r, \theta)$ (i.e. transforming from Cartesian to Polar and vice versa).

- Example (# 31, § 9.3)

$$x^2 - 4ay - 4a^2 = 0$$

$$(x,y) \to (r,\theta) \Rightarrow (r\cos\theta)^2 - 4a(r\sin\theta) - 4a^2 = 0 \Rightarrow \cos^2\theta\left[r^2 - \frac{4a\sin\theta}{\cos^2\theta}r\right] = 4a^2$$

$$\Rightarrow \cos^2\theta\left[r - \frac{4a\sin\theta}{\cos^2\theta}r + \frac{4a^2\sin^2\theta}{\cos^4\theta}\right] = 4a^2 + \frac{4a^2\sin^2\theta}{\cos^2\theta} \Rightarrow \cos^2\theta\left(r - \frac{2a\sin\theta}{\cos^2\theta}\right)^2 = 4a^2(1+\tan^2$$

(Completing Square)

$$\Rightarrow \cos^2\theta\left(r - \frac{2a\sin\theta}{\cos^2\theta}\right)^2 = 4a^2\left[\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\right] \Rightarrow \cos^2\theta\left(r - \frac{2a\sin\theta}{\cos^2\theta}\right)^2 = \frac{4a^2}{\cos^2\theta}$$

$$\Rightarrow \cos^2\theta\left(r - \frac{2a\sin\theta}{\cos^2\theta}\right)^2 = \frac{4a^2}{\cos^2\theta} \Rightarrow r - \frac{2a\sin\theta}{\cos^2\theta} = \pm\frac{2a}{\cos^2\theta} \Rightarrow r_{1,2} = \pm\frac{2a + 2a\sin\theta}{\cos^2\theta}$$

$$\Rightarrow r_{1,2} = \frac{\pm 2a(1 \pm \sin\theta)}{(1-\sin^2\theta)} \Rightarrow r_1(\theta) = \frac{2a(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)} = \frac{2a}{1-\sin\theta}$$

$$r_2(\theta) = \frac{-2a(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)} = \frac{-2a}{1+\sin\theta}$$

Hence there are two representations of the above, which we recognize as a parabola when put in standard form: $x^2 = 4a(y+a)$

$$\Rightarrow (x-0)^2 = 4a(y-(-a))$$

Center: (Vertex): $(0,-a)$ $p = a$

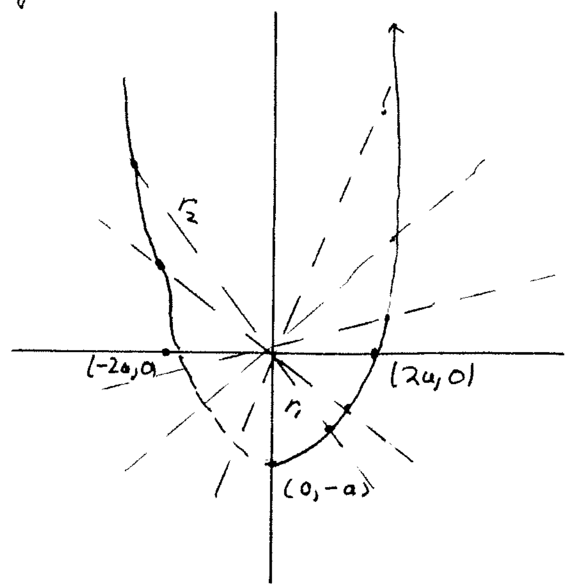
Observe: $r_1(\theta) = r_1(\pi-\theta)$ since $\sin\theta = \sin(\pi-\theta)$

also: $r_1(\theta) = -r_2(-\theta) = \frac{2a}{1-\sin(-\theta)} = \frac{2a}{1+\sin\theta}$ since $\sin(-\theta) = -\sin\theta$

But $r_1(\theta)$ "=" $r_2(\theta)$ (different parameterizations, but they represent the same graph) $\Rightarrow r(\theta) = -r(-\theta)$

So symmetric about y-axis

θ	$r_1(\theta)$	$r_2(\theta)$
$-\pi/2$	a	∞
$-\pi/3$	$\frac{4a}{2+\sqrt{3}}$	$\frac{-4a}{2-\sqrt{3}}$
$-\pi/4$	$\frac{4a}{2+\sqrt{2}}$	$\frac{-4a}{2-\sqrt{2}}$
0	$2a$	$-2a$
$\pi/6$	$4a$	$-\frac{4}{3}a$
$\pi/4$	$\frac{4a}{2-\sqrt{2}}$	$\frac{-4a}{2+\sqrt{2}}$
$\pi/3$	$\frac{4a}{2-\sqrt{3}}$	$\frac{-4a}{2+\sqrt{3}}$
$\pi/2$	∞	$-a$



So the parameterization $r_2(\theta)$ traces $r_1(\theta+\pi)$

which one can confirm: $|r_1(\theta+\pi)| = \frac{2a}{1-\sin(\theta+\pi)}$

$= \frac{2a}{1+\sin\theta} = |r_2(\theta)|$

27) $xy = 4$

$(r\cos\theta)(r\sin\theta) = 4 \Rightarrow \frac{r^2}{2}\sin 2\theta = 4 \Rightarrow r^2 = \frac{8}{\sin 2\theta} \Rightarrow r = \pm\frac{2\sqrt{2}}{\sqrt{\sin 2\theta}}$

or $r^2 = 8\csc 2\theta$

Note: (Recall p11 March 20 notes) $\cot 2\phi = \frac{A-C}{B} = \frac{0}{1} = 0 \Rightarrow 2\phi = \pi/2$

$\Rightarrow \phi = \pi/4$ (hyperbola rotated by $\pi/4$)

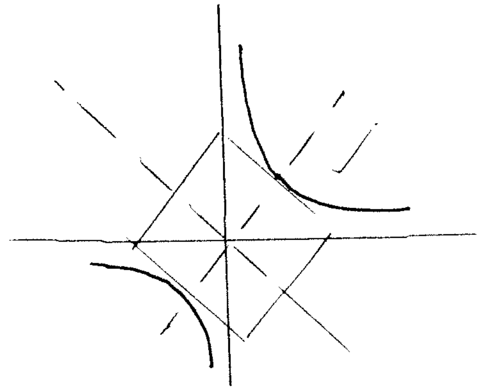
↑ Use phi notation, so as not to confuse it with polar angle θ

$x = x'\cos\phi - y'\sin\phi \Rightarrow x = \frac{\sqrt{2}}{2}(x'-y')$

$y = x'\sin\phi + y'\cos\phi \Rightarrow y = \frac{\sqrt{2}}{2}(x'+y')$

$$xy = \frac{\sqrt{2}}{2}(x'-y')\frac{\sqrt{2}}{2}(x'+y') = \frac{1}{2}(x'^2-y'^2) = 4 \Rightarrow x'^2 - y'^2 = 8$$

$$\Rightarrow \frac{x'^2}{(2\sqrt{2})^2} - \frac{y'^2}{(2\sqrt{2})^2} = 1$$



(19) $x^2+y^2-2ax=0$

$r^2 - 2ar\cos\theta = 0$

$r^2 - 2ar\cos\theta + a^2\cos^2\theta = a^2\cos^2\theta \Rightarrow (r - a\cos\theta)^2 = (a\cos\theta)^2$

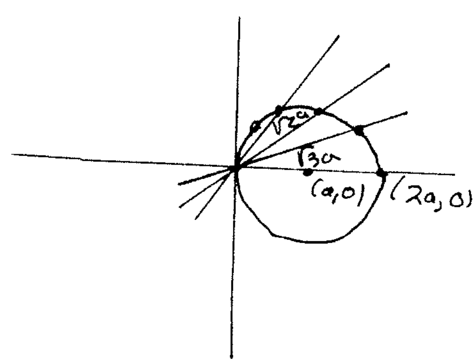
$\Rightarrow r - a\cos\theta = \pm a\cos\theta \Rightarrow r = 2a\cos\theta$

(one recognizes this as a circle, radius a, center $(a,0)$: $x^2-2ax+y^2=0$
$\Rightarrow x^2-2ax+a^2+y^2=a^2 \Rightarrow (x-a)^2+(y-0)^2=a^2$

Observe: $r(-\theta) = 2a\cos\theta = 2a\cos(-\theta) = r(\theta)$
or $-r(\pi-\theta) = -2a\cos(\pi-\theta) = -[-2a\cos\theta] = r(\theta)$
$\therefore$ symmetric about x-axis.

Moreover:

θ	r
0	$2a$
$\pi/6$	$\sqrt{3}a$
$\pi/4$	$\sqrt{2}a$
$\pi/3$	a
$\pi/2$	0



(32) $(x^2+y^2)^2 - 9(x^2-y^2)=0$

$(r^2)^2 - 9(r^2\cos^2\theta - r^2\sin^2\theta)=0$

$r^4 - 9r^2(\cos^2\theta - \sin^2\theta)=0 \Rightarrow r^2(r^2 - 9\cos 2\theta)=0 \Rightarrow r^2 = 9\cos 2\theta$

$r_{1,2} = \pm 3\sqrt{\cos 2\theta}$

(26) $4x + 7y - 2 = 0 \Rightarrow 4r(\cos\theta) + 7r\sin\theta = 2 \Rightarrow r(4\cos\theta + 7\sin\theta) = 2$

$\Rightarrow r(\theta) = \dfrac{2}{4\cos\theta + 7\sin\theta}$

(38) $r^2 = \sin 2\theta \Rightarrow r^2 = 2\sin\theta\cos\theta \Rightarrow r^4 = 2(r\sin\theta)(r\cos\theta) \Rightarrow (x^2+y^2)^2 = 2xy$

(39) $\theta = \pi/6 = \arctan(y/x) \Rightarrow \dfrac{y}{x} = \tan\pi/6 = \dfrac{1/2}{\sqrt{3}/2} = \dfrac{\sqrt{3}}{3} \Rightarrow \boxed{y = \dfrac{\sqrt{3}}{3}x}$

44) $r = \dfrac{6}{2\cos\theta - 3\sin\theta} \Leftrightarrow r^2 = \dfrac{6r}{2\cos\theta - 3\sin\theta} \Leftrightarrow r = \dfrac{6r}{2r\cos\theta - 3r\sin\theta} = \dfrac{6r}{2x-3y}$

$\Leftrightarrow 2x - 3y = 6$

(57) $r = 2(h\cos\theta + k\sin\theta)$

$r^2 = 2(hr\cos\theta + kr\sin\theta) = 2(hx + ky) \Rightarrow x^2 - 2hx + y^2 - 2ky = 0$

$\Rightarrow x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = h^2 + k^2$

$(x-h)^2 + (y-k)^2 = h^2 + k^2$

Circle: radius : $\sqrt{h^2+k^2}$ center (h,k)

(56) $d^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)$

$= (x_1^2 + y_1^2) + (x_2^2 + y_2^2) - 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$

$= r_1^2 + r_2^2 - 2r_1\cos\theta_1 r_2\cos\theta_2 - 2r_1\sin\theta_1 r_2\sin\theta_2$

$= x_1^2 + y_1^2 + x_2^2 + y_2^2 - 2x_1x_2 - 2y_1y_2$

$= x_1^2 - 2x_1x_2 + x_2^2 + y_1^2 - 2y_1y_2 + y_2^2 \Rightarrow (x_1-x_2)^2 + (y_1-y_2)^2$

Chain Rule (Thm 12.6)

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{d/d\theta\,(r(\theta)\sin\theta)}{d/d\theta\,(r(\theta)\cos\theta)} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}$$

where $r(\theta) = f(\theta)$ (some function of θ)

- Horizontal tangent line: $\frac{dy}{d\theta} = 0$ & $\frac{dx}{d\theta} \neq 0$

$\Rightarrow r'(\theta)\sin\theta + r(\theta)\cos\theta = 0$

& $r'(\theta)\cos\theta - r(\theta)\sin\theta \neq 0$

- Vertical tangent line: $\frac{dx}{d\theta} = 0$ & $\frac{dy}{d\theta} \neq 0$

$\Rightarrow r'(\theta)\sin\theta + r(\theta)\cos\theta \neq 0$

$r'(\theta)\cos\theta - r(\theta)\sin\theta = 0$

- Tangent lines at origin: For any $\theta = \alpha \Rightarrow r(\alpha) = 0$ but $r'(\alpha)$

then: $\frac{dy}{dx} = \frac{r'(\alpha)\sin\alpha}{r'(\alpha)\cos\alpha} = \tan\alpha \Rightarrow m = \tan\alpha$

or $\theta = \alpha$ tangent line at pole $(0,0)$

- Example (§ 12.4, # 9)

9) $r(\theta) = 2\csc\theta + 3$

$y(\theta) = r(\theta)\sin\theta = (2\csc\theta + 3)\sin\theta = 2 + 3\sin\theta$

$x(\theta) = r(\theta)\cos\theta = (2\csc\theta + 3)\cos\theta = 2\cot\theta + 3\cos t$

$y'(\theta) = \frac{d}{d\theta}(2 + 3\sin\theta) = 3\cos\theta = 0 \Rightarrow \theta_1 = \pi/2 \quad \theta_2 = 3/2\,\pi$

$x'(\theta) = -2\csc^2\theta - 3\sin\theta \Rightarrow x'(\theta_1) \neq 0, \; x'(\theta_2) \neq 0$

$\therefore \theta_n = \frac{\pi}{2} + n\pi = \pi(n + 1/2)$ yield tangent lines, horizontal

$r(\theta_1) = 2\csc\theta_1 + 3 = 5$

$r(\theta_2) = 2\csc\theta_2 + 3 = 1$

$\Rightarrow P_1(5, \pi/2 + 2n\pi)$

$P_2(1, \pi/2 + 2n\pi)$

(19) $r(\theta) = 3\cos 2\theta$

• Tangent at pole: $r(\alpha) = 0 = 3\cos 2\alpha \Rightarrow \cos 2\alpha = 0 \Rightarrow 2\alpha = \pm \frac{\pi}{2} \Rightarrow \alpha = \pm \frac{\pi}{4}$

$r'(\alpha) = -6\sin 2\alpha \Rightarrow r'(\alpha_{1,2}) \neq 0$

$\theta_1 = \pi/4 = \alpha_1$

$\theta_2 = -\pi/4 = \alpha_2$

Horizontal tangent: $x = r\cos\theta = 3\cos 2\theta \cos\theta$

$y = r\sin\theta = 3\cos 2\theta \sin\theta$

$\frac{dy}{d\theta} = r'(\theta)\sin\theta + r(\theta)\cos\theta$

$= -6\sin 2\theta \cdot \sin\theta + 3\cos 2\theta \cos\theta$

$= -12\sin^2\theta\cos\theta + 3(\cos^2\theta - \sin^2\theta)\cos\theta$

$= -12\sin^2\theta\cos\theta + 3\cos^3\theta - 3\sin^2\theta\cos\theta$

$= -15\sin^2\theta\cos\theta + 3\cos^3\theta$

$= -15(1-\cos^2\theta)\cos\theta + 3\cos^3\theta$

$0 = 18\cos^3\theta - 15\cos\theta \Rightarrow 6\cos^3\theta - 5\cos\theta = 0$

$\Rightarrow \cos\theta(6\cos^2\theta - 5) = 0 \Rightarrow \cos\theta_1 = 0 \Rightarrow \theta_1 = \frac{\pi}{2}$

$\cos^2\theta_2 = 5/6 \Rightarrow \theta_2 = \pm \arccos(\sqrt{5/6})$

$\frac{dx}{d\theta} = r'(\theta)\cos\theta + r(\theta)\sin\theta$

$= -6\sin 2\theta\cos\theta - 3\cos 2\theta\sin\theta$

$x'(\theta_1) \neq 0$ so θ_1 = horizontal tangent line pt.

$x'(\theta_2) \neq 0 \Rightarrow$

$-6\sin 2\theta_2 \sqrt{5/6} - 3[\cos^2\theta_2 - \sin^2\theta_2]\sin\theta_2$

$-12\sin\theta_2\cos\theta_2\sqrt{5/6} - 3[5/6 - 1/6]\sqrt{1/6}$

$-12\sqrt{1/6}\cdot(\sqrt{5/6})^2 - 2\sqrt{1/6} \neq 0$