Geometric-Algebraic Approaches to Quantum Physics: A Case for Ontological Unification

submitted by

William Michael Kallfelz

December 5, 2006

<u>Definition. 1 (Geometric Content):</u> The subclass of a physical theory's mathematical formalism that explicitly deals with a systematic characterization and analysis of entities like points, lines, planes, (hyper)surfaces and their respective interconnection, embedded in a space *X* of arbitrary dimension *n*.

<u>Definition</u>. <u>2 (Ontological Content):</u> The class of all metaphysically possible entities or representations within the theory's scope.

Remark 1. (Re: Geometric Content): Unless otherwise noted, X is taken to have the structure homeomorphic (i.e. topologically isomorphic) to a vector space of dimension n, equipped with metric $g_{\mu\nu}$ with signature (p,q), where $p + q = n$.

Remark 2a. (Re: Ontological Content): I select this notion of metaphysical possibility as a happy medium between the extremes of logical possibility (*P* is logically possible if not-*P* doesn't produce a contradiction) and nomological possibility (*P* is consistent with the body of truths expressed by the fundamental laws characterizing the physical theory).

<u>Remark 2b. (Re: Ontological Content):</u> The entities or representations in the theory should be taken as metaphysically primitive notions. I bracket out all such broader metaphysical issues concerning the nature of the relationship—whether weak, like reliability, or strong, like literal truth—between a theory's entities and the actual physical phenomena such entities in principle are meant to represent.

PSA

<u>Post-Standard Accounts (PSA)</u> (Frisch (2005))

Models are understood as providing an intermediary layer between a theory's laws and the "world" of phenomena. The existence of models can at best be understood as somewhat independent of the existence of a theory's laws.

"Building testable models, according to the [post-standard account] …usually involves highly context-dependent idealizing and approximating assumptions, and often requires appealing to assumptions from…sometimes incompatible theories." (10)

High-level abstractions, or the theory's "laws," should be thought of as "tools for model-building, rather than as representative of structures of the world." (11)

- **Note 1**: Remarks 2a., 2b. become substantiated in the richer ontology allowed by the PSA concerning a theory's models and other representative structures.

- **Note 2**: What I mean by unification by reduction in geometric and ontological contents shouldn't be conflated with Kitcher's unification account(s)—though there is conceptual overlap with theories of explanation (as in the case of my criticisms of Batterman (Kallfelz (2005b, 2006)). Though my project's primary aim doesn't involve issues of theories of scientific explanation per se.

- **A *foundational* aim**: Providing a coherent account of *the way possible worlds can be*, from which a set of fundamental laws governing phenomena in a certain domain can be derived.

- **A *pragmatic* aim** : Providing a practical and useful formalism, out of which laws can be *applied to model* specific phenomena.

Charity Principle

"[W]e should [examine]…which sets of equations physicists themselves take to be the most basic and important in a certain domain, and *then ask what criteria of theory-choice would allow us to make sense of the physicists' decisions*…[W]e should adopt a *principle of charity* and interpret the physicists' claims in a way that makes them defensible." (Frisch (2005) italics added, 70-72)

Flexibility Principle

"I think that some such distinction [between foundational versus pragmatic] is implicit in much philosophical theorizing about science…[though] *no conflict or trade-off between the foundational aim of providing a coherent account of what is physically possible and pragmatic considerations need arise*."
(Frisch (2005) italics added, 68)

Central Claim :

Clifford algebraic characterizations of quantum theory provide an interpretative framework in the sense of unifying to a greater extent (compared to the non-geometrical algebraic formalism) their geometrical and ontological content *both in a foundational and in a pragmatic sense.*

$$AB = A \bullet B + A \wedge B$$

Examples of Foundational Clifford Algebraic Projects:

- **<u>David Finkelstein (1999-2004)</u>** : Adopts a Clifford-algebraic characterization of a theory of a fully quantum space-time. The expressive power of a Clifford algebra is adequate to express a fully quantum theory of space-time geometry, where the *C** algebraic formalism would prove lacking. (Finkelstein, et. al. (2001), 5)

- <u>**Elio Conte (1993-2000)**</u> : Develops his biquaternionic quantum mechanics (or QM characterized by the Clifford Algebra $CL_{(4)}$) through which, for example, an interpretation of wave packet reduction (1993) and sketch of a 'new epistemology for quantum mechanics' (1996) is proposed. "The author emphasizes that…the formulation of a Biquaternion algebra…provides a unified and realistic language…giv[ing] the manner to generalize Bell's inequality and thus conceiving generalized quantum theories." (Conte (2000), 298)

- **<u>David Hestenes (1984, 1986)</u>** : Clifford algebras offer a reformulation of quantum theory associating any significant mathematical entity a semantic isomorphism with its algebraic and geometric meaning. For instance, in the case of complex vector spaces (i.e. a Hilbert space in standard QM mathematical formalism) "[o]ne can distinguish three fundamentally different geometric roles tacitly assigned to the unit imaginary $i = \sqrt{-1}$, namely:

(1) the generator of rotations in a plane [$\tilde{\varphi}(x,t) \mapsto e^{i\theta}\varphi(x,t)$]
(2) the generator of duality transformations [$\psi^{\uparrow}{}_{B} = \left(M_{AB}\psi^{A}\right)^{*}$ [1]]
(3) the indicator of an indefinite metric [$\psi^{\uparrow}\varphi = \left(\psi^{A}M_{AB}\psi^{B}\right)^{*}$, an indefinite sesquilinear form[2]]"

[1] "For complex action vectors, $\uparrow$ can be expressed as $\uparrow = M$, where is **complex conjugation** and M is a hermitian symmetric form." (Finkelstein (1996), 30.)

"Confusion is difficult to avoid when *i* is required to perform more that one of these roles in a single system. Worse yet, in physics all three possibilities are sometimes realized in a single theory confounded with problems of physical interpretation…[t]he multiplicity of geometric interpretations shows that conventional mathematical formalisms are profoundly deficient in their tacit assumption that there is a unique 'imaginary unit'. Therefore, in the interest of fidelity to geometric interpretation, the convention that complex numbers are scalars should be abandoned in favor of…a system in which each basic geometric distinction has a unique algebraic representation. Geometric [Clifford] Algebra has this property." (1984, xii – xiii).

---

[2] This is the definition of the metric ascribed to a *Dirac-space* , which can admit vector space decomposition: $_{U}\oplus$ - in which the Hilbert spaces $_{U}U'$ - are endowed with isometric (norm-preseerving) and anti-isomeric (**reversing the sign of the norm**) projections $_{U}U'$ - . (Finkelstein (1996), 552)

Clifford Algebras

- The Cambridge mathematician William Kingdon Clifford originally developed his algebra in the years 1878-1882 as a means to systematically develop a matrix algebra representing rotations and spin, generalized to any *n*-dimensional space: $R^n = \{(x_1, \ldots, x_n) \mid x_k \in R, 1 \le k \le n\}$ (where *R* are the real numbers).

- Geometric algebras hence can be fundamentally thought of as a systematic collections of *directed* line segments (vectors), areas (bivectors), volumes (trivectors),…, *n*-dimensional hypervolumes (*n*-vectors or *n*-blades) as bounded above by the dimensionality *n* of the algebra's underlying vector space.

- Clifford algebras are *graded*: their generators form a basis of linearly independent k-vectors (where $0 \le k \le n$), where n is the dimensionality of the underlying vector space. For example, the Clifford algebra $G(R^3)$ over vector space R^3 is generated by a total of $2^3 = 8$ grade k elements (where $0 \le k \le 3$): 1 grade-0 element (the real scalars), 3 grade-1 elements (3 linearly independent vectors whose span is obviously R^3), 3 grade-2 elements (3 linearly independent bivectors), and 1 grade-3 (trivector) element.

- In general, for any vector space V of dimensionality n, its Clifford algebra is generated by a total of 2^n grade k elements (where $0 \leq k \leq n$), the dimensionality of each Clifford subspace of uniform grade k is: $C(n,k)=\frac{n!}{k!(n-k)!}$. That is to say, $C(n,k)=\frac{n!}{k!(n-k)!}$ linearly independent grade-k (or k-vector) elements generate the Clifford subspaces of uniform grade k. In addition, the (associative) Clifford product can be decomposed into a grade-lowering (inner) product and a grade-raising (outer) product, from which the notions of dot and cross products survive in the standard (Gibbs) vector algebra of R^3.

- For any two elements A, B in a Clifford algebra CL, their Clifford product is defined by: $AB = A \bullet B + A \wedge B$, where $A \bullet B$ is their (commutative and associative) *inner* product, and $A \wedge B$ is their anti-commutative, i.e. $A \wedge B = -B \wedge A$, and associative *exterior* (or Grassmann) product. This naturally makes the Clifford product associative: $A(BC) = (AB)C \equiv ABC$. Less obviously, however, is how the *existence of an inverse* A^{-1} for every (nonzero) Clifford element A arises from the Clifford product, i.e.: $A^{-1}A = I = AA^{-1}$, where I is the *unit pseudoscalar* of CL.

- CL is equipped with an adjoint $^{\uparrow}$ and grade operator $\langle\ \rangle_r$ (where $\langle\ \rangle_r$ is defined as isolating the r th grade of a Clifford element A) such that, for any Clifford elements A, B: $\langle \mathrm{AB} \rangle^{\uparrow}_r = (-1)^{\mathrm{C}(r,2)} \langle \mathrm{B}^{\uparrow}\mathrm{A}^{\uparrow} \rangle_r$ (where: $\mathrm{C}(r,2) = {}^{r!}/_{(2!(r-2)!)} = {}^{r(r-1)/}{}_{2}$.)

- A general Clifford element (or multivector) A of Clifford algebra CL of maximal grade N = dimV (i.e the dimension of the underlying vector space structure of the Clifford algebra) is expressed by the linear combination:

$$A = \alpha^{(0)}A_0 + \alpha^{(1)}A_1 + \alpha^{(2)}A_2 + \ldots + \alpha^{(N)}A_N$$

where: $\{\alpha^{(\mathrm{k})} \mid 1 \le k \le N\}$ are the elements of the scalar field (expansion coefficients) while $\{A_\mathrm{k} \mid 1 \le k \le N\}$ are the *pure* Clifford elements, i.e. $<A_k>_l = A_k$ whenever $k = l$, and $<A_k>_l = 0$ otherwise, while for a general multivector $<A>_l = \alpha^{(l)}A_l$, for $1 \le l \le N$

- Consider the simple case of $V = \boldsymbol{R}^2$. Then; $N = \dim\boldsymbol{R}^2 = 2$. Moreover, $\boldsymbol{R}^2 = \langle(\hat{e}_1, \hat{e}_2)\rangle$, where $\langle\ldots\rangle$ denotes the *span* and $(\hat{e}_1, \hat{e}_2)$ are the ordered pair of orthonormal vectors (parallel, for example, to the *x* and *y* axes.) Hence: $\hat{e}_1{}^2 = \hat{e}_2{}^2 = 1$, and $\hat{e}_1 \bullet \hat{e}_2 = \hat{e}_2 \bullet \hat{e}_1 = 0$. So:

$$\hat{e}_1\hat{e}_2 = \hat{e}_2 \bullet \hat{e}_1 + \hat{e}_1 \wedge \hat{e}_2 = \hat{e}_1 \wedge \hat{e}_2 = -\hat{e}_2 \wedge \hat{e}_1 = -\hat{e}_2\hat{e}_1.$$ So:

$$(\hat{e}_1\hat{e}_2)^2 = (\hat{e}_1\hat{e}_2)(\hat{e}_1\hat{e}_2) = \hat{e}_1(\hat{e}_2\hat{e}_1)\hat{e}_2 = -\hat{e}_1(\hat{e}_1\hat{e}_2)\hat{e}_2 = -(\hat{e}_1\hat{e}_1)(\hat{e}_2\hat{e}_2) = -(\hat{e}_1^2)(\hat{e}_2^2) = -1$$

- Hence, the multivector $\hat{e}_1\hat{e}_2$ is algebraically isomorphic to $i = \sqrt{-1}$. Moreover, $(\hat{e}_1\hat{e}_2)\hat{e}_1 = -\hat{e}_2$ and $(\hat{e}_1\hat{e}_2)\hat{e}_2 = \hat{e}_1$. Geometrically, then, the multivector $\hat{e}_1\hat{e}_2$ when multiplying on the left has the effect of a *clockwise* $^{\pi}/_2$ –rotation.

- Represented then in the matrix algebra $M_2(\boldsymbol{R})$ (the algebra of real-valued 2x2 matrices):

$$\hat{e}_1\hat{e}_2 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{where: } \hat{e}_1 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \hat{e}_2 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- In $CL(\boldsymbol{R}^2)$ the multivector $\hat{e}_1\hat{e}_2$ is the *unit pseudoscalar,* i.e. the element of maximal grade.

- In general, for any Clifford Algebra $CL(V)$, where $\dim V = N$, and $V = \langle(\gamma_1, \gamma_2, \ldots, \gamma_N)\rangle$, where the basis elements aren't necessarily orthonormal, the unit pseudoscalar I of $CL(V)$ is: $I = \gamma_1\gamma_2\ldots\gamma_N$.

- For grade k (where $1 \le k \le N$) the closed subspaces $CL_{(k)}$ of grade k in $CL(V) = CL_{(0)} \oplus CL_{(1)} \oplus\ldots\oplus CL_{(\mathrm{N})}$ have dimensionality $\mathrm{C}(N,k) = {}^{N!}/_{[k!(N-k)!]}$, i.e are spanned by $\mathrm{C}(N,k) = {}^{N!}/_{[k!(N-k)!]}$ multivectors of degree k. Hence the total number of Clifford basis elements generated by the Clifford product acting on the basis elements of the underlying vector space is: $2^N = \sum_{k=0}^{N} C(N,k)$.

- The unit pseudoscalar is therefore the (one) multivector (only one there are $\mathrm{C}(N,N) = 1$ of them, modulo sign or order of mutliplication) spanning the closed Clifford subspace of maximal grade N.

- For example, in the case of $CL(\boldsymbol{R}^3) = CL_{(0)} \oplus CL_{(1)} \oplus CL_{(2)} \oplus CL_{(3)}$, where:

$\boldsymbol{R}^3 = \langle(\hat{e}_1, \hat{e}_2, \hat{e}_3)\rangle$: $CL_{(0)} = \langle 1 \rangle \cong R, CL_{(1)} = \langle(e_1, e_2, e_3)\rangle, CL_{(2)} = \langle(e_{12}, e_{13}, e_{23})\rangle, CL_{(3)} = \langle I \rangle = \langle e_{123} \rangle$

(where the abbreviation $e_{\mathrm{i \ldots k}} = \hat{e}_i \ldots \hat{e}_k$ is adopted).

- The unit psuedoscalar I should *not* be interpreted as a multiplicative identity, i.e. it is certainly *not* the case that for any $A \in CL(V)$, $AI = A = IA$. Rather, the unit pseudoscalar is adopted to define an element of dual grade A^* : for any pure Clifford element $A_{k,}$ (where $0 \leq k < N$) : the grade of AI (or A^*) is $(N- k)$, and vice versa. Thus an inverse element A^{-1} can in principle be constructed, for every nonzero $A \in CL(V)$. So the linear equation $AX = B$ has the formal solution $X = A^{-1}B$ in $CL(V)$. "Much of the power of geometric (Clifford) algebra lies in this property of invertibility." (Lasenby, et. al. (2000), 25)

Example: Spinors (*Hestenes* (*1986*))

"Physicists generally regard the σ_k [Pauli spin matrices] as three components of a single vector, instead of an orthonormal frame of three vectors…Consequently, they write: $\vec{\sigma} \bullet \vec{v} = \sum_{k=1}^{3} \sigma_k v_k$ …and to facilitate manipulation they employ the identity:

$$(\vec{\sigma} \bullet \vec{v})(\vec{\sigma} \bullet \vec{w}) = \vec{v} \bullet \vec{w} + i\vec{\sigma} \bullet (\vec{v} \times \vec{w})$$

… a good example of the redundancy in the language of physics which complicates the manipulations and obscures the meanings unnecessarily." (1986, 323)

The redundancy in the above identity is due to its ‘overlapping geometric content’: The (vector) dot and cross products of course comprise the binary operations of standard (Gibbs’) vector algebra in R^3, while the Pauli spin matrices $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ acting as the ‘vector coefficients’ belong to the spinor algebra C^2, (the 2×2 matrix algebra consisting of complex-valued entries).

- The geometric contents of R^3 and C^2 can be unified, however, when one instead considers σ_k as generators of a Clifford algebra, thereby "eliminat[ing] all redundancy incorporating both languages into a single coherent language."

- To see this, simply write for any 3-vector $\vec{v} = \sum_{k=1}^{3} v_k \sigma_k$, then the above identity with its (otherwise geometrically overlapping content) now simply is represented as: $\vec{v}\vec{w} = \vec{v} \bullet \vec{w} + \vec{v} \wedge \vec{w}$. This is just precisely the definition of the Clifford product of two 3-vectors.

- Using the same reasoning as demonstrated above in the case of the Pauli spin algebra, Hestenes (1986) likewise shows how the 4×4 matrix algebra C^4 of Dirac spinors γ_μ is algebraically isomorphic to the Clifford algebra C_4, or the Clifford algebra generated by 4-dimensional vectors with complex coefficients.
- This Clifford algebra is projectively isomorphic to the Minkowski spacetime algebra $R_{1,3}$, or the Clifford algebra generated by four linearly independent rotation matrices in the Minkowski spacetime $R_{1,3}$.

“The relation of the Dirac theory [of spinors] to classical electrodynamics is not well understood…[with the projective extension into $R_{1,3}$ however] it is more intimate than originally thought….

This intimate relation between …the Dirac theory and trajectories of the classical theory [shown in the $R_{1,3}$ reformulation] provides a much more detailed correspondence between the classical and quantum theories than the conventional approach using expectation values and Ehrenfest’s theorem…[T]he basic idea…we have been exploiting provides a general geometrical approach to the interpretation of the Dirac theory as follows…any solution $\psi = \psi(x)$ of the Dirac Equation of form $\psi = (\rho e^{i\beta})^{1/2}R$ [where ρ, is a probability density, R is a spinor representation of a Lorentz transformation Λ, and β is an arbitrary phase factor] determines a field of orthonormal frames $e_\mu = e_\mu(x)$…at each spacetime point there’s a streamline $x = x(\tau)$” with tangent $v = v(x(\tau))$. [Then] $e_\mu = e_\mu(x(\tau))$ is to be regarded as a ‘comoving frame’, on the streamline, where e_1 , e_2 rotate about the ‘spin axis’ e_3 .” (332-333)

- From the classical solution of the Dirac Equation, Hestenes derives the result: $\dot{R} = \frac{d}{d\tau} R = \frac{e}{2m} FR$, or the (electron's proper time) rate of change of R (the spinor representation of the Lorentz transformation Λ in the solution of canonical form $\psi = (\rho e^{i\beta})^{1/2} R$) as proportional to the product of the electron's charge with R and the magnitude of the electromagnetic field F.

- Hestenes interprets $\dot{R} = \frac{d}{d\tau} R = \frac{e}{2m} FR$ as an expression of the *precession* of the comoving frame $e_\mu = e_\mu(x(\tau))$, with an additional rotation determined by a gauge factor. "It should be of genuine physical interest to identify and analyze any deviations from this classical rotation which QM might imply." (333) Hestenes (1985, 13) suggests that such expressions of precession provide adequate models for the supposed *Zitterbewegung* mechanism of a free electron.

"My objective…has been to explicate the geometric structure of the Dirac theory and its physical significance. My approach may seem radical at first sight, but…it [is] ultimately conservative…by restricting my mathematical language to spacetime algebra [i.e. the Clifford algebra over Minkowski space $R_{1,3}$] I allow nothing in my formulation of physical theory without an interpretation of spacetime geometry…[though] I am not opposed to investigating possibilities for unifying physical theory by extending spacetime geometry to higher dimensions…we still have a lot to learn about the physical implications of conventional spacetime structure." (346)

"The most important thing…from the [Clifford algebraic] reformulation [of the Dirac theory] is that the imaginary $i = \sqrt{-1}Id$ [where *Id* is the identity operator] has definite geometrical and physical meaning…represent[ing] the generator of rotations in a spacelike plane associated with spin… $i = \sqrt{-1}Id$ can be identified with the spin bivector $\hat{s} = i\hbar\vec{\sigma}$ …[This identification] has far reaching consequences…[for instance] when the Schroedinger equation is derived as an approximation to the Dirac equation…[this] implies that a degenerate representation of the spin direction by the unit imaginary has been implicit in Schroedinger equation all along." (331)

$$i = \sqrt{-1}Id$$

From a foundational point of view, the common weak point of all proposals for quantum spacetimes reside in their lack of a clear-cut theory of measurement, that could point the way to reliable and empirically well founded geometric ideas. Such ideas could have either enabled the injection of the fundamental principles of relativistic invariance, causality and locality into these frameworks, or might have alternatively produced compelling principles that would transcend into the macroscopic regime, where…SR and GR have received most convincing experimental confirmation. (Eduard Prugovečki[3], (1992). p 5)

---

[3] *Quantum Geometry* : *A Framework for Quantum General Relativity*. Fundamental Theories of Physics, vol. 48, Dordrecht: Kluwer Academic Publishers.

RBQM

- Relational Blockworld Non-Relativistic QM

  Michael Silberstein, W. M. Stuckey, Michael Cifone (2006a-f)[4]

- Acausal kinematic ("Archimedean Geometric") interpretation of NRQM

[4] Forthcoming: *IJTP* (2006a), *On the Dimensionality of the World*, v. d. Merwe, ed. (2006b), *SHPMP : Special issue-Time Symm QM* (2006c), IQSA 2006 (2006d), Found Prob. (Sweden) (2006e), "Relational Blockworld: Radically Archimedean Physics" (2006f)

NRQM $\rightarrow$ RQFT

- To what extent can the geometric content of RBQM be enriched with Clifford algebraic characterizations of QFT?
- What ontological ramifications would this pose for RBQM? For instance, would the kinematic/dynamic distinctions remain unaffected? Why or why not?
- Would a Clifford Algebraic characterization of RBQM substantiate my Claim 2? In other words, would the exercise further simplify RBQM's ontological and geometrical content? Why or why not?