Ultraviolet Divergences, Renormalization, and Anomalies in Quantum Mechanics

Wm. D. Linch, III

May 3, 2008

Abstract

I review and elaborate on an example due to Mead and Godines [1] of regularization and renormalization in quantum mechanics.

1 The bound state of the attractive $\delta$-function potential

The time-independent Schrödinger wave equation for a particle of mass $M$ and energy $E$ in a potential $V(x)$ is

$$-\frac{\hbar^2}{2M} \nabla^2 \psi(x) + V(x)\psi(x) = E\psi(x).$$  \hfill (1)

We will work in $D = 2 + 1$ dimensions and study the attractive potential of a fixed point particle at $x = 0$ coupling to our particle with strength $\lambda > 0$

$$V(x) = -\lambda \delta^{(2)}(x).$$  \hfill (2)

Note that this potential is classically non-singular in the sense that particle motion is well defined on $\mathbb{R}^2 \setminus \{0\}$.

This system enjoys the rigid spacial symmetry $\mathbb{C}^* \approx U(1) \times \mathbb{R}$ of rotations around the origin of coordinates and scaling symmetry

$$x \mapsto \Omega x$$  \hfill (3)

for $\Omega > 0$ a constant, provided $E \mapsto \Omega^{-2} E$.  

1
We begin by considering solutions to the Schrödinger equation with $E < 0$ called bound states. Define the quantities
\[ g_0 = \frac{2M\lambda}{\hbar^2} \Rightarrow [g_0] = 1 \]
\[ m^2 = \frac{2M(-E)}{\hbar^2} \Rightarrow [m] = L^{-1} \] (4)
and rearrange the Schrödinger equation to the modified Helmholtz equation (with source)
\[ (\nabla^2 - m^2)\psi(x) = -g\delta^{(2)}(x)\psi(x). \] (5)

We solve this equation by Green's method. Introduce the Green function $G(x, y)$ as fundamental solution to the equation
\[ (\nabla_x^2 - m^2)G(x, y) = -\delta^{(2)}(x - y). \] (6)
The use of such a form is that the "solution" of the modified Helmholtz equation (5) for arbitrary source $\rho(x)$ is given by
\[ \psi(x) = \psi_0(x) - \int d^2yG(x, y)\rho(y) \] (7)
where $\psi_0$ is a solution of the sourceless Helmholtz equation. In our equation the source is $\rho(x) = -g_0\delta^{(2)}(x)\psi(x)$ and we want the solution to decay rapidly at $\infty$ for square-integrability. The latter requirement implies that $\psi_0$ is a solution of the modified Bessel equation which vanishes at $\infty$ and is regular everywhere. Contrary to the case of the Bessel equation, this implies that $\psi_0 \equiv 0$.

The equation for the Green function (6) can be solved by Fourier transformation
\[ G(x - y) = \int \frac{d^2q}{(2\pi)^2} e^{iq \cdot (x-y)} \tilde{G}(q). \] (8)
Plugging in gives
\[ \tilde{G}(q) = \frac{1}{q^2 + m^2} \] (9)
and, hence,
\[ G(x, y) = \int \frac{d^2q}{(2\pi)^2} \frac{e^{iq \cdot (x-y)}}{q^2 + m^2} = \frac{1}{(2\pi)^2} \int_0^\infty dq \frac{q}{q^2 + m^2} \int_0^{2\pi} d\theta e^{iq|x-y|\cos \theta} \] (10)
where \( q = |q| \). “Recalling” the integral formula for the Bessel function \( J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(x \cos \theta - n\theta)} d\theta \) we can rewrite this as

\[
G(x, y) = \frac{1}{2\pi} \int_0^\infty dq \frac{qJ_0(q|x-y|)}{q^2 + m^2} = \frac{1}{2\pi} K_0(m|x-y|).
\] (11)

Here I have cheated and used formula 6.532.4 from Gradshteyn and Ryzhik [2] for the modified Bessel function of the second kind \( K_0(x) \). The behavior of this function near 0 and \( \infty \) is

\[
K_0(x) \sim \begin{cases} 
-\log \frac{x}{2} - \gamma & \text{for } 0 < x \ll 1 \\
\sqrt{\frac{2}{\pi x}} e^{-x} & \text{for } x \gg \frac{1}{4}
\end{cases}
\] (12)

In particular, our Green function diverges logarithmically as \(|x - y| \to 0\).

The integral equation (7) with Green’s function (11) for the Helmholtz operator and with the source \( \rho \) given by the potential \( V \) is completely equivalent to the original Schrödinger equation (1). The bound-state solution of Schrödinger’s equation is then

\[
\psi(x) = \frac{g_0}{2\pi} \psi(0) K_0(m|x|).
\] (13)

Consistence of this equation implies that \( m = \infty \), that is, the energy \( E \) diverges.

### 1.1 Regularization

The quantum mechanics problem described above is ill-defined. We can make it well defined by cutting off the short-distance divergence. This can be done

---

1. We will be using some of the Bessel functions \( J_\alpha(x) \) and \( Y_\alpha(x) \) of the first and second kind, the modified Bessel functions \( I_\alpha(x) \) and \( K_\alpha(x) \) of the first and second kind, and the Hankel functions \( H^{(1)}_\alpha(x) \) and \( H^{(2)}_\alpha(x) \). These functions are the analogues of the trigonometric, hyperbolic, and exponential functions. We represent the analogy in the form of a table:

<table>
<thead>
<tr>
<th>1-dimensional</th>
<th>2-dimensional</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cos(x) )</td>
<td>( J_\alpha(x) )</td>
</tr>
<tr>
<td>( \sin(x) )</td>
<td>( Y_\alpha(x) )</td>
</tr>
<tr>
<td>( e^{ix} )</td>
<td>( H^{(1)}_\alpha(x) )</td>
</tr>
<tr>
<td>( e^{-ix} )</td>
<td>( H^{(2)}_\alpha(x) )</td>
</tr>
<tr>
<td>( e^x )</td>
<td>( I_\alpha(x) )</td>
</tr>
<tr>
<td>( e^{-x} )</td>
<td>( K_\alpha(x) )</td>
</tr>
</tbody>
</table>
in a way which preserves rotation invariance at the expense of scale invariance: Let us replace the classical problem of an attractive point charge at the origin by an attractive spherical particle at the origin with radius \( \epsilon > 0 \). For notational (in)convenience we define \( \Lambda = 1/\epsilon \). This gives the regularized potential analogous to (2)

\[
V_\epsilon(x) = -\lambda \frac{1}{r} \delta \left( r - \frac{1}{\Lambda} \right) \delta(\theta).
\] (14)

The quantum mechanical problem is now well-defined and the equation for the bound state wave function (13) is changed to

\[
\psi(x) = \frac{g_0}{2\pi} \psi(\Lambda^{-1}) K_0(m|x|).
\] (15)

Consistency now gives finite bound-state energy which according to the short distance behavior of the modified Bessel function (12) is

\[
E_{bs} = -\frac{\hbar^2}{2M} 4e^{-2\gamma} \Lambda^2 e^{-\frac{4\pi}{g_0}}.
\] (16)

The energy is a physical parameter which is directly measurable in experiments\(^2\). It’s value should not diverge as the cutoff is removed \( \Lambda \to \infty \). This is apparently impossible. The solution is to observe that we do not measure the bare coupling \( g_0 \) directly. Instead we measure quantities like the scattering cross section (below) which depend implicitly on the bare coupling. For example, we can define the energy to be independent of the cutoff by allowing the bare coupling to have the dependence

\[
\frac{1}{g_0(\Lambda)} = -\frac{1}{2\pi} \log \left( \frac{m}{2\Lambda} \right) - \frac{\gamma}{2\pi}
\] (17)

where we keep \( m \) fixed to some value we will measure in our experiment.

Despite the fact that we never measure the coupling directly, it is useful to define a “renormalized coupling” \( g_r \) which is independent of the cutoff \( \Lambda \). This is done by subtracting the divergent part \( \sim \frac{1}{2\pi} \log \Lambda \) of the bare coupling. Since the cutoff is a dimensionful parameter, it is not possible to do this (we cannot take the logarithm of a dimensionful parameter) without introducing another with the same dimensions. This parameter is called the “subtraction point” and is traditionally denoted by \( \mu \). We then define the renormalized coupling by

\[
\frac{1}{g_r(\mu)} = \frac{1}{g_0(\Lambda)} + \frac{1}{2\pi} \log \left( \frac{\mu}{2\Lambda} \right) + \frac{\gamma}{2\pi}.
\] (18)

\(^2\)For example one can hit the bound state with a laser of frequency \( \omega \) and vary this frequency. There will be a peak in the emission of the mass \( M \) particles at the “resonance” \( \omega = \omega_0 \equiv E_0/\hbar \).
The renormalized coupling has the property that it remains finite as the cutoff $\Lambda \to \infty$ because we have subtracted the divergent part $\frac{1}{2\pi} \log \Lambda$. The dependence of the coupling on the subtraction point is encoded in the celebrated “β-function” defined by the Gell-Mann-Low equation

$$\beta(g_r) \overset{\text{def}}{=} \mu \frac{d}{d\mu} g_r(\mu) = -\frac{1}{2\pi} g_r^2(\mu).$$

(19)

where we have used the formula $\mu \frac{d}{d\mu} \frac{1}{g_r(\mu)} = \frac{1}{2\pi}$. The solution to the β-function equation is

$$\frac{1}{g_r(\mu)} = \frac{1}{g_r(m)} + \frac{1}{2\pi} \log \left( \frac{\mu}{m} \right).$$

(20)

Note that the subtraction in (18) is quite arbitrary. To wit, we could have added any finite dimensionless terms to the definition of the renormalized coupling. In fact, we have conveniently chosen this constant so that the energy (16) of the bound state can be expressed in terms of the renormalized coupling and the subtraction point as

$$E_{bs} = -\frac{\hbar^2}{2M} \mu^2 e^{-4\pi/g_r(\mu)}.$$  

(21)

Written in this form, the energy is manifestly finite in the limit $\Lambda \to \infty$. It appears to depend on the arbitrary subtraction point, which is impossible since the latter is arbitrary. In fact, this expression for the bound state energy satisfies the “Callan-Symanzik equation”

$$0 = \mu \frac{d}{d\mu} E_{bs} = \left[ \mu \frac{\partial}{\partial \mu} + \beta(g_r) \frac{\partial}{\partial g_r} \right] E_{bs}$$

(22)

expressing this independence.

1.2 Scattering in two dimensions

Above we studied the (unique!) bound state for the attractive regularized δ-function potential. To get more observables, we consider the scattering of free particles off this same potential. All this requires is to take the energy $E > 0$ to be positive. In this case, the analogue of $m^2$ becomes $k^2 = 2mE/\hbar^2$. Note that with this definition $p = \hbar k$ is the particle’s momentum. Note also that all formulæ above still apply with the formal replacement $m^2 \to -k^2$. This turns the modified Helmholtz operator $\nabla^2 - m^2$ into the Helmholtz operator $\nabla^2 + m^2$. The kernel of the latter are the represented in Cartesian coordinates by plane waves $e^{ik \cdot x}$ with $k^2 = m^2$ and in polar coordinates by Bessel functions.
The Bessel functions \( J_\alpha(x) \) and \( Y_\alpha(x) \) are 2-dimensional analogues of cosines and sines respectively. The analogue of the complex exponentials are the Hankel functions \( H^{(1)}_\alpha(x) = J_\alpha(x) + iY_\alpha(x) \) and \( H^{(2)}_\alpha(x) = J_\alpha(x) - iY_\alpha(x) \). The out-going radial wave should have the phase structure \( e^{ikr} \) which corresponds to the Hankel function of the first kind.

The Green function for the Helmholtz equation becomes

\[
G(x, y) = \frac{i}{4} H^{(1)}_0(k|x - y|)
\]

where \( H^{(1)}_0(x) \) is the lowest Hankel function that solves Bessel’s equation (related to the modified Bessel equation by \( m^2 \to -k^2 \)) with asymptotic behavior

\[
H^{(1)}_0(x) \sim \begin{cases} 
1 + \frac{2i}{\pi} \left[ \log \frac{x}{2} + \gamma \right] & \text{for } 0 < x \ll 1 \\
\sqrt{\frac{2}{\pi x}} e^{i(x-\pi/4)} & \text{for } x \gg \frac{1}{4}
\end{cases}
\]

Asymptotically the outgoing radial waves are therefore \( \sim \frac{1}{\sqrt{r}} e^{ikr} \).

The quantum mechanical wave function representing the scattering of an incoming plane wave off of a central potential is therefore quite generally of the asymptotic form

\[
\psi(x) \sim A \left\{ e^{ikx} + f(k, \theta) \frac{e^{i(kr-\pi/4)}}{\sqrt{r}} \right\}
\]

with the “momentum” of the out-going radial wave \( k = |k| \) equal in magnitude to that of the incoming wave as required by conservation of momentum. The scattering amplitude \( f(k, \theta) \) encodes the probability for scattering in the direction \( \theta \): The probability that an incoming particle passing through a cross-sectional “area” \( d\sigma \) will scatter into a “solid angle” \( d\Omega = d\theta \) is \( d\sigma = |f(k, \theta)|^2 d\Omega \). That is, the differential cross section

\[
\frac{d\sigma}{d\Omega} = |f(k, \theta)|^2.
\]

The integrated cross section \( \sigma \) is the integral of the differential cross section over all solid angles, indicating that a particle passing through this cross section will be scattered in some direction. That is, the target has an effective cross sectional size \( \sigma \). In three dimensions a cross section has the units of area and in two, the units of length. Naive application of dimensional analysis using the bare parameters of the original formulation of the scattering problem implies that for small coupling

\[
\sigma(k) = c \frac{q^2}{k^3}
\]
for some constant $c$ of order 1. We already know, however, that this cannot be correct since the bare coupling is renormalized. Replacing it with the renormalized coupling does not help since this quantity depends on the subtraction point. We will see in the calculation below that the actual cross section as computed in the renormalized theory is quite different from this naïve expectation. When expressed in terms of the renormalized coupling we will find that it obeys a Callan-Symanzik equation $\mu \frac{d}{d\mu} \sigma = 0$.

The solution to the scattering problem for the wave function is again given as in (7) but with this new Green function. In the scattering context equation (7) is referred to as the Lippmann-Schwinger equation

$$
\psi(x) = \psi_0(x) - \int d^2y G(x, y) \rho(y) = e^{ik \cdot x} + \frac{ig_0}{4} \psi(\Lambda^{-1}) H_0^{(1)}(k|x|).
$$

(28)

Here we have included the vacuum solution $\psi_0(x) \sim e^{ik \cdot x}$ to satisfy the boundary conditions of an incoming particle.

Again, we must impose consistency of this scattering solution at $|x| = \Lambda^{-1}$. The result is that the wave function at the cutoff scale is given by

$$
\psi(\Lambda^{-1}) = \frac{1}{1 - g_0 G(\Lambda^{-1})}
$$

(29)

where $G(\Lambda^{-1}) = \frac{i}{4} - \frac{1}{2\pi} \left[ \log \frac{k}{2\Lambda} + \gamma \right]$ is the regularized Green function at the cutoff. The asymptotic expansion of the Hankel function (24) implies that the regularized scattering amplitude

$$
f(k, \theta) = \frac{ig_0}{\sqrt{8\pi k}} \psi(\Lambda^{-1}) = \frac{i}{\sqrt{8\pi k}} \frac{1}{\frac{1}{g_0} - G(\Lambda^{-1})}.
$$

(30)

This expression is superficially divergent. However, at this point, we recall that what we mean by the coupling $g_0$ is actually regularized coupling $g_0(\Lambda)$ (17). In fact, in terms of the renormalized coupling (18)

$$
\frac{1}{g_0(\Lambda)} - G(\Lambda^{-1}) = \frac{1}{g_r(\mu)} + \frac{1}{2\pi} \log \frac{k}{\mu} - \frac{i}{4}
$$

(31)

Plugging this in, we obtain the renormalized scattering amplitude

$$
f(k, \theta) = i \sqrt{\frac{\pi}{2k}} \frac{1}{g_r(\mu)} + \log \frac{k}{\mu} - \frac{i\pi}{2}.
$$

(32)

The differential scattering cross section is

$$
\frac{d\sigma}{d\Omega} = \frac{2\pi}{k^2} \frac{1}{\pi^2 + \log^2 \left( \frac{k^2}{m^2} \right)}.
$$

(33)
Since there is no angular dependence to this formula, we can integrate over \( d\Omega = d\theta \) to obtain the total cross-section. In terms of the scattering energy \( E \) this gives

\[
\sigma(E) = \frac{4h}{\sqrt{2ME}} \frac{\pi^2}{\pi^2 + (\log E - \log |E_{bs}|)^2}
\]  

(34)

1.3 Physics of the scattering amplitude

The scattering amplitude is a function of the form

\[
y(x) = \frac{1}{\sqrt{x}} \frac{a^2}{a^2 + (\log x)^2}
\]  

(35)

The second factor is a Lorentzian curve peaking with height \( y = 1 \) at \( x = 1 \) and having width \( a \). The \( x^{-\frac{1}{2}} \) factor makes the curve lean to the left. Clearly \( \lim_{x \to \infty} y(x) = 0 \). Two applications of l'Hôpital’s rule suffices to conclude that \( \lim_{x \to 0} y(x) = +\infty \). In between the function may or may not have critical points. Since \( (y^{-1})' = -y^{-2}y', \) \( y(x) \) and \( 1/y(x) \) have extrema at the same points. Taking the derivative of \( 1/y \), multiplying \( 2\sqrt{x}a^2 \), and setting the result to 0 gives a quadratic equation in the logarithm. The solutions to this equation are

\[
\log x = -2 \pm \sqrt{4 - a^2}.
\]  

(36)

Therefore, for \( 0 < a < 2 \) the curve \( y(x) \) starts at \( +\infty \) comes down to a local minimum at \( x = e^{-2-\sqrt{4-a^2}} \), rises to a local maximum at \( x = e^{-2+\sqrt{4-a^2}} \), and finally asymptotes to 0 as \( x \to \infty \).

The physics of this behavior is the following. Starting in the high energy regime \( E \to \infty \) the target is invisible to the probe. This corresponds to high momentum, which in turn corresponds to short wavelength \( \lambda \to 0 \), and is called the deep ultraviolet (UV) regime. The scattering in this regime is called hard scattering. Contrary to what the name suggests, the probe actually does not scatter off the potential at all and behaves instead as a free particle. The is called asymptotic freedom.

As we come down from the ultraviolet, the cross-section grows until it achieves a maximum somewhere around \( E \lesssim |E_{bs}| \). This enhancement of the interaction at the bound state energy is called a resonance. The width of the peak gives a characteristic time scale called the lifetime of the resonance and allows the interpretation of this process as the formation of an unstable particle.

Proceeding into the infrared regime (IR) the cross-section decreases again to a non-zero value. This is the effective size of the target due to quantum
mechanics effects as seen by any low-energy (a.k.a. soft) probe. We note that this effective size is non-zero even thought the classical cross section of a point particle is identically zero.

Finally, in the deep infrared, the cross-section suddenly diverges very rapidly. In quantum electrodynamics, this effect is called the Coulomb singularity. The enhancement of the cross-section in the deep IR is a physical effect which comes from the fact that a particle with very small kinetic energy has almost no speed and, therefore, “lingers” in the interaction region for a long time. This divergence, as all IR divergences, comes from the infinitely long time a 0-momentum particle has to interact with the target. In a realistic experiment the particle detector cannot see probes with an energy lower than some fixed energy $E_{\text{res}}$ called the detector resolution (actually, inverse of). Therefore, to see the Coulomb singularity one would need a perfect detector.

This physics is all very nice. Unfortunately for us $a = \pi$. If $a = 2$ the local extrema merge into an inflection point and for $a > 2$ there is no nice peak leaving only a “crossover region” around $x \sim 1/e^2$. This point is called the weak resonance. We therefore find that the cross-section behaves more boringly as

$$\sigma(E) \sim \begin{cases} 0 & \text{as } E \to \infty \\ \infty & \text{as } E \to 0 \end{cases}$$

leaving only asymptotic freedom and the Coulomb singularity as features.

## 2 General Principles

In the quantum mechanical problem of a probe particle scattering off a fixed (or very heavy), point-like target we encountered divergences. Although this theory is of a very specific type (i.e. single particle, $D = 2 + 1$, non-relativistic, etc.) it turns out that the structure of the divergences are of the form of those of a relativistic quantum field theory. Indeed, formally replacing the Laplacian $\nabla^2 \to \Box$ with the d’Alembertian, we change the Helmholtz operator into the Klein-Gordon operator. Happily, therefore, we can see the general physical consequences of divergences, for which quantum field theories are famous (or notorious), already in this simple example. We collect the salient points here.

---

3This changes the interpretation of the parameter $m$ so that it becomes the mass of the probe. Similarly, the coupling $g_0$ becomes the fine structure constant $\alpha_0$ (which is basically the coupling squared). Regardless of interpretation, the program of renormalization goes through practically unmodified.
2.1 Renormalizability

A theory often depends for its definition on parameters (or even functions) which do not correspond to physical observables. In the example we just studied, it was necessary to start with a coupling constant to write the Schrödinger equation describing the physics even though there is no physical observable corresponding directly to this parameter. (Of course, in this description, observables may depend on this parameter.) It can then happen that calculations in such a theory are divergent. The question becomes (1) whether it is possible to absorb all divergences in these unphysical parameters, and (2) whether the physical observables are independent of this “renormalization” process.

Suppose the definition of a theory requires the use of \( m \) bare parameters \( \{g_0_i\}_{i=1}^m \). The observables, forming a set \( \{O_{0\alpha}\}_{\alpha=1}^n \), will depend on these parameters \( O_{0\alpha}(g_0_i) \). One proceeds to compute expectation values of these operators and their products and finds that they are divergent. The renormalization to be carried out proceeds in 4 steps [1]:

1. Regularize: Make all (ultraviolet) divergences finite by one universal parameter \( \Lambda \) called the regulator. The operators now depend on this cutoff \( O_{0\alpha} = O_{0\alpha}(g_0_i, \Lambda) \).

2. Choose \( m \) operators and fix them to measured values. Solve the equations \( O_{0\alpha} = O_{0\alpha}(g_0_i, \Lambda) \) for the parameters \( \{g_0_i\}_{i=1}^m \) in terms of these values. This makes the couplings cutoff-dependent \( g_0_i = g_0_i(\Lambda) \). These are the regularized couplings.

3. In the rest of the \( n - m \) operators \( \{O_{0\alpha}\}_{\alpha=m+1}^n \), substitute the cutoff-dependent values of the couplings. These operators are now finite and referred to as regularized.

4. Remove the cutoff \( \Lambda \rightarrow \infty \).

If all operators in the set \( \{O_{0\alpha}\}_{\alpha=1}^n \) remain finite in step [4] the original theory has been successfully renormalized. If there is no scheme for which this procedure works, the theory is said to be non-renormalizable.

4The term “renormalization” seems to come from the fact that one of the unphysical functions in quantum field theory is the wave function itself (the others are usually masses and couplings). This means that the procedure requires the re-normalization of this function as the cutoff is changed.

5Of course these do not have to be actually measured values from an experiment; we are merely fixed some subset of all operators.
2.2 The running coupling

In the problem above we defined the $\beta$ function \[19\], reproduced here

$$\beta(g_r) \overset{\text{def}}{=} \mu \frac{d}{d\mu} g_r,$$

(38)
as the change in the renormalized coupling under a rescaling of the subtraction point. We computed this function after performing step 2 in the renormalization procedure by fixing the bound state energy.

For $\beta \neq 0$ the renormalized coupling depends on the arbitrary subtraction point and can, therefore, not correspond to a physical quantity. (How would you measure it? Probably by scattering!) Nevertheless, it is an extremely intuitive device as it tells us how strong the interaction is. Indeed, the renormalized amplitude \[32\], and, therefore, the differential cross-section, vanishes as $g_r \to \infty$ while keeping the probe momentum fixed.

Once we begin to consider scattering experiments, we perform introduce a new scale into the problem, namely, the momentum $p = \hbar k$ of the probe. As the subtraction point was arbitrary to begin with, we may conveniently choose it to be $\mu = k$. The $\beta$-function equation now implies that the renormalized coupling changes with the momentum of the probe according to

$$p \frac{d}{dp} g_r = \beta(g_r).$$

(39)

When the coupling is considered a function of the probe momentum as it is here, one refers to it as the running coupling.

Suppose the $\beta$-function of a given theory is positive at some probe momentum $p$. Then as we increase the momentum, the coupling increases. The physical cartoon of this is as follows: The probe can penetrate deeper into the interaction region (i.e. get closer to the target) when it has smaller wavelength and hence higher momentum. As it does this it sees an effective increase in the charge of the target. This effect is called the screening of the charge of the target and comes from the polarization of the vacuum around the charge. This is what happens in abelian gauge theories and quantum electrodynamics.

Now suppose that, as is the case here, the $\beta$-function is negative. Then the coupling runs down as we increase the probe energy. If, as in our case, the $\beta$-function stays negative, the coupling asymptotes to 0. That is, the theory

---

6 In that same formula, we gave the exact result for the $\beta$-function for this theory. In a more general theory this $\beta$-function is computed, if possible, perturbatively in the bare coupling. Nevertheless, the first non-trivial contribution to the $\beta$-function is often that shown in equation \[19\].

7 This assumes the $\beta$-function is analytic in the coupling, a natural assumption if the theory is formulated perturbatively. This could be violated by “non-perturbative” effects, that is, effects that go like $1/g$. Then this happens, the coupling could hit the $\mu$ axis.
is non-interacting in the deep UV. This behavior is reflected in the vanishing of the cross-section at high energies as we found above. This phenomenon is called *asymptotic freedom* and is the hallmark of non-abelian gauge theories and quantum chromodynamics.

Finally, suppose that at some value \( g^* \) the \( \beta \)-function vanishes \( \beta(g^*) = 0 \). Then the coupling stops running at \( g = g^* \). The theory at such a point stays there and, in addition, has an invariance under a change of scale \( \mu \rightarrow \Omega^{-1}\mu \). Such a point is, therefore, called a *conformal fixed point*. A quantum field theory with \( \beta \equiv 0 \) is called a *conformal field theory* (CFT).

### 2.3 Scale anomaly

We started with a classical theory which had a scaling symmetry [3]. The naïve quantization of this theory resulted in ultraviolet divergences. The renormalization procedure produced a quantization of the classical theory in the formal sense discussed in class. This quantum point-source problem, however, is definitely *not* scale invariant as we can clearly see from the scattering cross-section:

\[
\sigma(\Omega^{-2}E) \neq \Omega \sigma(E).
\]

This phenomenon is referred to as an *anomaly* and the scaling symmetry is said to be anomalous. A universal feature of scale anomalies is that they have a chance to arise because of the appearance of ultraviolet divergences. Had the calculation of the bound state energy above been UV-finite, the coupling would not run and the scattering amplitude would have depended on the probe energy according to naïve dimensional analysis [27].

### 2.4 Strong coupling

In general, we are not lucky enough to be able to solve a theory exactly. We can still attempt to apply the techniques described here perturbatively in the renormalized coupling provided the coupling is “small enough” for some region in momentum space. Solving the Lippmann-Schwinger equation perturbatively gives an expansion of the \( \beta \)-function in powers of the coupling. For example, in an asymptotically free theory the renormalized coupling is small at high momentum transfer and the \( \beta \)-function is negative at leading order. As we bring the momentum down, the coupling grows until at some point \( p \sim \Lambda_0 \) (this notation is historical and \( \Lambda_0 \) should not be confused with the regulator \( \Lambda_0 \)!) it becomes of order

\[
g_r(\Lambda_*) \sim 1, \quad (40)
\]

and perturbation theory breaks down completely. It is important to distinguish this scale \( \Lambda_* \) which is *fixed* to some value by this equation from the
renormalization group parameter $\mu$.

Although the renormalized coupling is, strictly speaking, not a physical observable, what is happening here is the failure of the entire approximation scheme, that is, the definition of the quantization of the classical theory breaks down. Again, although the actual (unknown) quantum theory must remain well-defined, our description of it must be altered in the strong-coupling regime. Now the description we use to define the theory depends on physical observables and therefore, the breakdown of perturbation theory corresponds to a physical change in these observables.

Now, we started with a dimensionless parameter $g_0$ and ended up with a renormalized parameter $g_r(\mu)$ which satisfies the equation (40). This implies that we can replace everywhere the dimensionless coupling with the dimensionful parameter $\Lambda$. Thus, the physics is now parameterized by a new physical scale with the dimensions of $L^{-1}$. This effect of the ultraviolet divergences is called dimensional transmutation.

Note added I also found this great paper by Jackiw [4] in which he describes the more formal mathematical basis for the renormalization scheme as the failure of the Hamiltonian to be Hermitian. His regularization method is by defining a “self-adjoint extension” of the Hamiltonian.

References


