Notes on Maldacena duality

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1 Quantum Field Theory

1.1 Gauge fields

Connection $\nabla = d + A$ on principle $\text{SU}(N)$-bundle over 4-dimensional Minkowski space $(\mathbb{R}^4, \eta)$ with curvature $F = \frac{i}{2} [\nabla, \nabla]$.

Yang-Mills action $S_{\text{YM}} \sim \frac{1}{g_{\text{YM}}} \int d^4x \text{Tr} F^2 + \frac{\vartheta}{4\pi^2} \int F \wedge F$.

Symmetries:

- Global: Scale $x \mapsto e^{\Omega} x$ in $\text{D}=4$
- Local: Gauge $\nabla \mapsto g \nabla g^\dagger$ and Weyl $\eta_{ab} \mapsto e^{2\omega(x)} \eta_{ab}$ in $\text{D}=4$
- Duality

Observables: Gauge invariant combinations $\mathcal{O}$ of $A$ e.g. $\mathcal{O} \sim \text{Tr}(F \nabla \nabla ... F ... E E ...)$

Correlation functions $\langle \mathcal{O}_1 ... \mathcal{O}_n \rangle = \int \mathcal{O}_1 ... \mathcal{O}_n e^{-S_{\text{YM}}}[dA]$ e.g. Path integral $Z \sim \int e^{-S_{\text{YM}}}[dA] = \langle 1 \rangle$

"Solving QCD" (pure Yang-Mills in this case) means find a basis of "observables" $\{\mathcal{O}\}$ and all correlation functions $\langle \mathcal{O} ... \mathcal{O} \rangle$.

Problems:

- Gauge symmetry $\Rightarrow$ ghosts

---

1. Actually, full conformal invariance. The conformal group $\text{Spin}(4,2) \approx \text{SU}(2,2)$ contains the Poicaré group $\text{Spin}(3,1) \times \mathbb{R}^4$ generated by Lorentz transformations and translations $\{M_{ab}, P_a\}$ extended by conformal and scale transformations $\{K_a, \Delta\}$.

2. The path-ordered exponential $P \exp M(t) = 1 + \int_0^t dt' M(t') + \int_0^t dt' \int_0^{t'} dt'' M(t') M(t'') + ...$ is the solution to the equation

\[
\frac{\partial}{\partial t} P \exp M(t) = M(t) P \exp M(t)
\]

subject to the initial condition $P \exp M(0) = 1$. 
• UV divergences ⇒ Regularization ⇒ Renormalization ⇒ Running coupling
  (Scale anomaly) ⇒ no perturbation theory at all energies

• Number of Feynman diagrams at order $n$ is of order $n!(\text{const})^n$

1.1.1 Toy model for quantum effects

Map $n : \mathbb{R}^2 \to S^{N-1}$

Action for massless, interacting field

$$ S \sim \frac{1}{2g_0^2} \int d^2x \sqrt{\gamma} \gamma^{ab} \partial_a n_i(x) \partial_b n^i(x) : n_i(x) n^i(x) = 1 $$

Symmetries:

- Global $O(N)$ and scale $x \mapsto e^{i\Omega} x$
- Local Weyl $\gamma_{ab} \mapsto e^{2\omega(x)} \gamma_{ab}$

Lagrange multiplier field $m(x)$

$$ Z \sim \int [dn] e^{-S} \delta(n^2 - 1) \sim \int [dn][dm] e^{-\frac{1}{2g_0^2} \int m(n^2 - 1)} $$

makes path integral Gaußian which we can integrate to

$$ Z \sim \int [dm] \frac{g_0^N}{\sqrt{\det \Delta^m}} e^{\frac{1}{2g_0^2} \int m} \sim \int [dm] e^{-S'} $$

with action $S' = -\frac{N}{2} \int d^2x \left[ -\frac{m}{g_0^2 N} + \log \det \Delta^m \right]$ and Laplacian $\Delta^m = -\Box + m(x)$.

Large $N$ approximation: $N \to \infty$ and $g \to 0$ such that $\lambda_0 = g_0^2 N$ fixed

Localization of $Z$ to stationary point of $S'$: $m_* \text{ s.t. } \frac{\delta S'}{\delta m(x)} \bigg|_{m=m_*} = 0$

Fourier transform and regulate with regulator $\Lambda$:

$$ \frac{\partial}{\partial m} \log \det' \Delta^m = \frac{\partial}{\partial m} \log \prod_{p \in \mathbb{R}^2} (p^2 + m) = \int \frac{dp}{(2\pi)^2} \frac{1}{p^2 + m} = \frac{1}{4\pi} \log \frac{\Lambda^2}{m} $$

Vacuum expectation value

$$ m_* = \Lambda^2 e^{-\frac{4\pi}{\lambda_0}} $$
implies mass \( m_n^2 = m_* \) generated (non-perturbatively) for \( n(x) \)

\[
S[n, m = m_*] \sim \frac{1}{2g_0^2} \int d^2x \sqrt{\gamma} \left[ \gamma^{ab} \partial_a n_i \partial_b n^i + m_n^2 n_i n^i + \text{const.} \right]
\]  

Mass \( m_n \) is physical (measured in laboratory) \( \Rightarrow \) independent of regularization \( \Rightarrow \) coupling not a parameter

\[
\lambda(\mu) = -\frac{1}{2\pi} \log \frac{m_n}{\Lambda}
\]

Renormalized coupling: Subtract divergent part \( \lambda(\mu) = \lambda(\Lambda) + \frac{1}{2\pi} \log \frac{\mu}{\Lambda} \)

Quantum effects:

- Scaling and Weyl symmetries anomalous
- Running coupling: \( \mu \frac{d}{d\mu} \lambda = \beta(\lambda) \) with \( \beta(\lambda) = -\frac{1}{2\pi} \lambda^2 \)
- Asymptotic freedom: \( \lambda \to 0 \) as \( \mu \to \infty \)
- ... Symmetry restoration

### 1.1.2 Feynman diagrams

Map \( M : \mathbb{R}^4 \to \text{SL}_N \)

Action \( S \sim \frac{1}{g_0^2} \int \text{Tr}[(\partial M)^2 + V(M)] \)

Partition function \( Z \sim \int [dM] e^{-S} \)

Propagator \( \sim g_0^2 \)

Vertices \( \sim g_0^{-2} \)

Loops \( \sim N \)

“Fatgraph” Feynman diagram with \( v \) vertices, \( e \) edges, and \( f \) faces \( \sim (g_0^2)^{-v} N^f \)

- Euler number \( \chi = v - e + f = 2 - 2g \) on Riemann surface of genus \( g \)
- 't Hooft coupling \( \lambda = g_0^2 N \)
- Diagrams \( (g_0^2)^{-v} N^f = N^\chi \lambda^{f-\chi} \)
- 't Hooft large \( N \) limit: \( N \to \infty \) and \( g_0 \to 0 \) with \( \lambda \) fixed

Effective action localizes on planar diagrams

\[
S_{\text{eff}} = -\log Z \sim \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda) \longrightarrow N^2 f_0(\lambda) + O(1)
\]

When \( \lambda \gg 1 \), the sum over planar diagrams is dominated by \( f \to \infty \) \( \Rightarrow \) closed sphere with punctures
2 Gravity

2.1 Strings

Map \( x : (\Sigma, \gamma) \to (\mathbb{R}^D, \eta) \)

Polyakov action:

\[
S_x \sim \frac{1}{2\ell^2} \int d^2\sigma \sqrt{\gamma} \gamma^{\mu\nu} \partial_\mu x^a \eta_{ab} \partial_\nu x^b
\]  

Symmetries:

- Global target: Poincaré: \( x \mapsto \Lambda x + a \)
- Local worldsheet: \( \text{Diff}(\Sigma) \) and Weyl \( \gamma_{\mu\nu} \mapsto e^{2\omega(\sigma)}\gamma_{\mu\nu} \)

New term with same symmetries:

\[
\chi \sim \frac{1}{4\pi} \int d^2\sigma \sqrt{\gamma} R(\gamma)
\]  

Physics determined by symmetries \( \Rightarrow S = S_x + \Phi_0\chi \)

Partition function\(^3\):

\[
Z \sim \sum_\Sigma \int [dx][d\gamma] e^{-S_x} e^{-\Phi_0\chi} \sim \sum_{g=0}^\infty (e^{-\Phi_0})^{2-2g} f_g(\ell)
\]  

0-loop \( \sim e^{-2\Phi_0} \), 1-loop \( \sim 1 \), ... Closed string loop expansion parameter: \( g_s^2 = e^{2\Phi_0} \)

Closed string loop expansion has same form as ’t Hooft large-\( N \) expansion with

\[
g_s \leftrightarrow \frac{1}{N} \sim g_{YM}^2
\]

Suggests finding string theory such that string tree level \( \leftrightarrow \) planar diagrams\(^4\)

String action invariant under diffeomorphisms and local Weyl transformations

\(^3\)Use 1PI effective action instead.

\(^4\)As pointed out to me by Brenno Vallilo, this vision was never fully realized. String theories exhibit open-closed duality which implies \( g_c = g_o^2 \) (attaching strips to open string worldsheets reduces the Euler number by 1 instead of 2), that is \( g_s = g_{YM}^2 \) and not \( 1/N \), as in ’t Hooft’s large-\( N \) expansion. So a purist would say the string expansion is an expansion in \( \lambda/N \) and therefore it is not the ’t Hooft expansion.
Use diff to fix metric $\gamma_{ab} \rightarrow e^{\phi(\sigma)}\delta_{ab}$
Requires reparameterization ghosts $(b, c)$
Define
\[ e^{-F[\gamma]} \sim \int [dx][db][dc] e^{-S_x[x,\gamma] - S[b,c,\gamma]} \] (13)

Now shift $\gamma \rightarrow e^{\phi(\sigma)}\gamma$

Claim:
\[ F[e^{\phi}\gamma] - F[\gamma] = \frac{26 - D}{48\pi} \int d^2\sigma \left[ \frac{1}{2} (\partial\phi)^2 + \phi R(\gamma) + \mu^2 e^\phi \right] \] (14)

Liouville action: particle with potential wall $\sim e^\phi$
Therefore, if we start in $D<26$, we grow a warped extra dimension

Actually, cannot quantize Liouville theory in $D>1$, nevertheless we guess that gauge theory in $D=4$ corresponds to a string theory in $D=5$ with a warped extra dimension

Most general metric:
\[ ds^2 = w^2(z) \left( dx^2 + dz^2 \right) \] (15)

### 2.2 Backgrounds

Replace flat background with NLSM
\[ S \sim \frac{1}{2\ell^2} \int d^2\sigma \sqrt{\gamma} \left[ \gamma^{\mu\nu}G_{ab}(x)\partial_\mu x^a\partial_\nu x^b + e^{\mu\nu}B_{ab}(x)\partial_\mu x^a\partial_\nu x^b + \frac{\ell^2}{2\pi} \Phi(x) R(\gamma) \right] \] (16)

Linearized approximation: $G_{ab}(x) = \eta_{ab} + h_{ab}(x)$, $B_{ab}(x) = b_{ab}(x)$ and $\Phi(x) = \Phi_0 + \varphi(x)$

Symmetries:
- Infinitesimal diffeomorphism of target space $\delta h_{ab} = \partial_a \xi_b + \partial_b \xi_a$ (using equation of motion for $x$)
- U(1) gerbe relation $\delta b_{ab} = \partial_a \alpha_b - \partial_b \alpha_a$

This is the gauge invariance of massless, spin-2 particle suggesting gravitation on target space
Einstein equation $R_{ab}(G)$ for $G$??
figure out how to motivate masslessness, Einstein equation, D=26...

Bosonic string $\beta$-functions:

\[
\beta_{G_{ab}} = \ell^2 R_{ab} + 2\ell^2 \nabla_a \nabla_b \Phi - \frac{\ell^2}{4} H_{acd} H_{b}^{cd} + O(\ell^4)
\]

\[
\beta_{B_{ab}} = -\frac{\ell^2}{2} \nabla^c H_{abc} + \ell^2 \nabla^c \Phi H_{abc} + O(\ell^4)
\]

\[
\beta_\Phi = \frac{D-26}{6} - \frac{\ell^2}{2} \nabla^2 \Phi + \ell^2 \nabla \Phi \cdot \nabla \Phi - \frac{\ell^2}{24} H^2 + O(\ell^4)
\]

The first equation is the Ricci flow, modified by a diffeomorphism. It is the $L^2$ gradient of the effective action:

\[
S_{\text{eff}} \sim -\frac{1}{2\kappa_0^2} \int \chi \left[ \Box + R + \frac{1}{12} H^2 \right] \chi + \ldots
\]

where $\chi^2 = \sqrt{G} e^{-\Phi}$ is the T-duality invariant dilaton (see below).

Local Weyl invariance: $\Phi \mapsto \Phi + D\omega$ and $G_{ab} \mapsto e^{2\omega} G_{ab}$

String frame: $\omega = 0$. Einstein frame: $\Phi = \Phi_0$

Gravitational coupling $\kappa = \kappa_0 e^{\Phi_0} = \sqrt{8\pi G_N} = \sqrt{8\pi \ell_p^{D-2}}$

3 Putting it all together

The classical YM theory has the scale invariance $x \mapsto e^{\Omega_0} x$

This has to be a symmetry of the string $\Sigma$-model action $\Rightarrow$ symmetry of warped metric (15) $\Rightarrow$ warp factor $w(z) = \frac{L}{z}$ for some constant $L$ setting the length scale:

\[
ds^2 = L^2 \frac{dx^2 + dz^2}{z^2}
\]

This is the $AdS_5$ metric in Poincaré coordinates.

Problems:

- Yang-Mills theory is not conformal quantum mechanically
- Easy strings live in $D=26$ (bosonic) and $D=10$ (supersymmetric) not $D=5$
- $AdS_5$ has $R_{ab} = -(5L)^{-2} g_{ab} \neq 0$. I.e. not a string background unless $\partial \Phi \neq 0$

These problems cancel out:
• Conformal invariance restored with maximal supersymmetry: $\mathcal{N} = 4$

• Replace spacetime with $AdS_5 \times X$ where $\text{dim}(X) = 11$ or 5

• Choose $\text{dim}(X) = 5$ and curvature $R_{ab|X} = +(5L)^{-2}g_{ab}$, i.e. $X = S^5$ with radius $L$ Fix this: Only the scalar curvature vanishes.

We seek a superstring s.t. the low-energy supergravity on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ super-Yang-Mills with gauge group $SU(N)$ in the large-$N$ limit.

3.1 Anti-de-Sitter spaces

Double Wick rotation: $S^n \to H^n \to AdS_n$

Homogeneous and isotropic Einstein space:

$$AdS_n \approx Spin(n-1,2)/Spin(n-1,1)$$  \hspace{1cm} (20)

Isometries $Spin(4,2) \approx SU(2,2)$ fixing

$$-X_1^2 - X_0^2 + X_1^2 + \ldots + X_{n-1}^2 = L^2$$  \hspace{1cm} (21)

Poincaré coordinates $z^{-1} = L^{-2}(X_1 + X_{n-1})$ and $(x)_a = z^{-1}LX_a$

Induced metric

$$ds^2 = \frac{L^2}{z^2}(dz^2 + dx^2)$$  \hspace{1cm} (22)

3.2 Maximally supersymmetric Yang-Mills theory

Super-Poincaré algebra $\{P_a, Q_{\alpha A}, \bar{Q}_{\dot{\alpha} A}, M_{ab}, R_{A B}\}$ with $4 \times 4 = 16$ real supercharges

Lorentz:

$$[M, Q] \sim Q, \ [M, P] \sim P$$  \hspace{1cm} (23)

Supersymmetry:

$$\{Q, \bar{Q}\} \sim P$$  \hspace{1cm} (24)

$SU(4)_R$-symmetry:

$$[R, Q] \sim Q$$  \hspace{1cm} (25)
Special conformal generator\(^5\) \(K_a \sim x^2 \partial_a - 2x_a x \cdot \partial\) extends:

\[
[K, P] \sim M + \Delta
\]  
(27)

Dilatation (scale transformation) generator \(\Delta \sim x \cdot \partial\):

\[
[\Delta, Q] \sim \frac{1}{2} Q , \quad [\Delta, P] \sim P , \quad [\Delta, K] \sim -K
\]  
(28)

Special conformal generators map supercharges to “\(S\)-supersymmetry” charges

\[
[K, Q] \sim \bar{S} , \quad \{S, \bar{S}\} \sim K , \quad \{Q, S\} \sim M + \Delta
\]  
(29)

Conformal generators double the number of fermionic generators from 16 to 32.

Superconformal group \(PSU(2,2|4) \supset SU(2,2) \times SU(4) \approx Spin(4,2) \times Spin(6)\):

\[
\text{psu}(2, 2|4) \sim \begin{pmatrix}
M & P + iK & Q + i\bar{S} \\
- P - iK & \Delta & R \\
- Q - iS & - R & \end{pmatrix}
\]  
(30)

Field representation (all in adjoint of \(SU(N)\)):

\[
\varphi^{AB} \overset{Q_a}{\longrightarrow} \lambda^A \overset{Q_A}{\longrightarrow} dA
\]  
(31)

On-shell supersymmetry \(\Rightarrow\) 6 real scalars \(\varphi^{AB} = -\varphi^{BA} = \frac{1}{2} \epsilon^{ABCD} \bar{\varphi}_{CD}\)

In superspace \(W^{AB} \sim \varphi^{AB} + \theta^A \lambda^B + \theta^A \sigma^{ab} \theta^B F_{ab}\)

Action:

\[
S \sim \text{Tr} \int d^4\theta \, \tau W^2 + \text{h.c.}
\]  
(32)

Complexified coupling: \(\tau = \frac{\vartheta}{2\pi} + i \frac{2\pi}{g_{YM}}\)

Symmetry: \(\tau \mapsto \tau + 1\) and \(S\)-duality \(\tau \mapsto -\frac{1}{\tau}\) generate \(SL(2,\mathbb{Z})\)

\(^5\)The finite action of a special conformal transformation is the combination inversion-translation-inversion:

\[
K_a x^b \sim IP_a Ix^b \sim I\partial_a \frac{x^b}{x^2} \sim I \frac{1}{x^2} \left( \delta^b_a - \frac{2x_a x^b}{x^2} \right) \sim x^2 \left( \delta^b_a - \frac{2x_a x^b}{x^2} \right) \sim (x^2 \partial_a - 2x_a x \cdot \partial) x^b
\]  
(26)
3.3 Type IIB superstrings

The superstring we seek should have the properties

- live in D=10 i.e. critical superstring
- 32 supersymmetries i.e. “type II”
- S-duality $\text{SL}(2,\mathbb{Z})$

This string exists and has left- and right-moving spinors of the same chirality

IIB Ramond-Ramond field-strengths:

$$s \otimes s = \bigoplus_{n \text{ odd}} v \wedge \ldots \wedge v \Rightarrow F[1], F[3], F^+_{[5]}$$  \hspace{1cm} (33)

Potentials $C[0], C[2], C[4]$ couple to instantons, strings and self-dual 3-branes
Together with NS-NS fields $h_{ab}, b_{ab}, \varphi \Rightarrow$ possibility of $\text{SL}(2,\mathbb{Z})$ duality:

$$\tau = \frac{\vartheta}{2\pi} + i \frac{2\pi}{g_Y^2} \leftrightarrow C[0] = \frac{C[0]}{2\pi} + 2\pi i e^{-\Phi_0}$$  \hspace{1cm} (34)

Consistency check: Are dilaton and RR scalar constant in $\text{AdS}_5 \times S^5$ background?

Action in Einstein frame:

$$S \sim \frac{1}{2\kappa^2} \int \text{d}^{10}x \sqrt{G} \left[ R(G) + \frac{1}{2} \sum F^2 \right]$$  \hspace{1cm} (35)

Below we will see that $\int_{S^5} F[5] \sim N$. Then EOM $\Rightarrow R^{10-2} \sim N^2$:

$$R \sim (g_s N)^{\frac{1}{4}} \ell_s \sim N^{\frac{1}{4}} \ell_p$$  \hspace{1cm} (36)

4 Black branes

Einstein equation:

$$\text{Ric} - \frac{1}{2} gR = 8\pi GT$$  \hspace{1cm} (37)

Schwarzschild black hole a.k.a. “black 0-brane” has $T_{00} \sim M \delta(r)$:

$$ds^2 = -\left(1 - \frac{r_S}{r}\right) dt^2 + \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 + r^2 d\Omega^2_2$$  \hspace{1cm} (38)
Schwarzschild radius: \( r_S = 2GM \) with \( M \) the gravitational charge \( \int T_{00} \, d\text{vol} \sim M \)

Penrose diagram:

Charged Reissner-Nordström black hole \( A_0 = Q/r \):

\[
\begin{aligned}
\text{ds}^2 &= -(1 - \frac{r_+}{r})(1 - \frac{r_-}{r}) \, dt^2 + (1 - \frac{r_+}{r})^{-1} (1 - \frac{r_-}{r})^{-1} \, dr^2 + r^2 d\Omega_2^2 \\
\text{Roots} \quad r_{\pm} &= \frac{1}{2} \left( r_S \pm \sqrt{r_S^2 - 4r_Q^2} \right) \text{of horizon condition} \quad 0 = g_{00} = 1 - \frac{r_-}{r} + \frac{r_+^2}{r^2} \\
\text{Charge radius} \quad r_Q = \sqrt{G}Q \text{ with } Q \text{ the electromagnetic charge} \quad \int_{S^2} F \sim Q \\
\text{Cosmic censorship:} \quad r_- \leq r_+ \Rightarrow Q \leq \sqrt{GM} \\
\text{Extremal black hole (no Hawking radiation) is BPS} \quad Q = M: \text{ “D0-brane”} \\
\text{Near-horizon limit} \quad y = r - r_S \to 0 \\
\text{ds}^2 &\to -\frac{y^2}{R^2} dt^2 + \frac{R^2}{y^2} \, dy^2 + R^2 d\Omega_2^2 \\
\text{with} \quad R = 2GM = 2\sqrt{G}Q. \text{ Poincaré coordinates} \quad z = R^2/y \\
\text{ds}^2 &\to R^2 \frac{-dt^2 + dz^2}{z^2} + R^2 d\Omega_2^2 \\
\text{Near-horizon geometry of extremal black 0-brane in D}=4 \text{ is } \text{AdS}_2 \times S^2 \text{ with radius } R \text{ determined by the amount of flux through the 2-sphere.}
\end{aligned}
\]

5 Correspondence

Supergravity approximation \( R \gg \ell_s \Rightarrow \lambda \gg 1 \) valid for strong gauge theory coupling.
Fix radius \( R = 1 \Rightarrow \ell_s^2 \sim 1/\sqrt{g_sN} \) and \( \kappa^{-2} \sim \ell_p^{-8} \sim N^2 \).

Gravitational partition function:

\[
Z[\phi \sim \phi_0] = e^{-N^2S_{\text{classical}} + O(\ell_s^2)} \left( 1 + O(g_s^2) \right)
\]

Boundary condition \( \phi'(x, z) \sim_{\infty} \phi_0(x) \) corresponds to an \( \mathcal{N} = 4 \) super-Yang-Mills operator \( O_i \).

Witten’s prescription: \( Z[\phi \sim \phi_0] = \langle e^{i \int \phi \, O_i} \rangle \)

For large ’t Hooft coupling \( \lambda \gg 1 \) the generating functional for SYM:

\[
\langle e^{i \int \phi \, O_i} \rangle \approx e^{-N^2S_{\text{classical}}[\phi \sim \phi_0]} \\
\]

How to determine \( O_i \)?
5.1 Example: Scalar field

Take $\phi$ a scalar field of mass $m$.

Action:

$$S \sim N^2 \int d^4x \, dz \frac{1}{z^5} \left[ z^2 (\partial \phi)^2 + m^2 R^2 \phi^2 + O(\phi^4) \right] \quad (44)$$

Classical action requires stationary point (EOM):

$$z^3 \frac{\partial}{\partial z} \left( \frac{1}{z^2} \frac{\partial \phi}{\partial z} \right) - p^2 z^2 \phi - m^2 R^2 \phi = 0 \quad (45)$$

Near boundary $\phi \sim z^\alpha$:

$$\alpha (\alpha - 4) - m^2 R^2 = 0 \quad \Rightarrow \quad \alpha_\pm = 2 \pm \sqrt{4 + m^2 R^2} \quad (46)$$

Dominant solution:

$$\phi(x, z) \sim e^{\alpha_-} \phi_0(x) \quad (47)$$

Scaling symmetry $(x, z) \mapsto e^{\Omega_0}(x, z)$ $\Rightarrow$ scaling dimension of $\phi_0$ is $\alpha_-$. Then Witten’s prescription implies

$$\Delta_\phi = 2 + \sqrt{4 + m^2 R^2} \quad (48)$$

For dilaton $\Phi$, $m = 0 \Rightarrow \Delta_\phi = 4$ which suggests $\mathcal{O}_\phi$ is the SYM Lagrangian.