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Geometric Algorithms for Computing Cutter Engagement Functions in 2.5D Milling Operations

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ABSTRACT

Cutter engagement is a measure that describes what portion of the cutter is involved in machining at a given instant of time. During profile milling of complex geometries, cutter engagement varies significantly along the cutter path. Cutter engagement information helps in determining the efficiency of the cutter path and also helps in improving it by adjusting the feed rate. This paper describes geometric algorithms for computing piece-wise continuous closed-form cutter engagement functions for 2.5D milling operations. The results produced by our algorithm are compared with the results obtained by discrete simulations of the cutting process and appear to match very well.

KEYWORDS: Cutter Engagement, Feed Rate Adjustments, Cutter Path Planning.

1 INTRODUCTION

Cutter engagement is a measure that describes what portion of the cutter is actually involved in machining at a given instant of time. Figure 1 shows an illustration of cutter engagement during milling. During complex milling operations, usually only a portion of cutter engages in cutting and, therefore the cutter engagement varies along the cutter path. The variation in cutter engagement results in corresponding variation in the cutting force. Therefore, by monitoring the cutter engagement, we can monitor and control cutting forces. A sudden increase in cutter engagement may even result in tool breakage. Determination of cutter engagement is essential for adjusting feed rate [Jerard et al. 1986, Spence et al. 1990, Fussell et al. 2001]. Cutter engagement value can also be used in generating efficient cutter paths [Yao and Gupta 2004].

Two different approaches are possible for determining cutter engagement. The first type is based on discrete sampling techniques [Jerard et al. 1986]. The second type is based on analytical techniques and provides continuous closed-form cutter engagement functions [Spence et al. 1990 Spence and Altintas 1991, Kramer 1992, Spence et al. 2000]. Analytical approaches for computing cutter engagement give more accurate values of cutter engagement compared to sampled points in the case of discrete sampling techniques. The numerical method has the limitation that it cannot handle very large stock parts and at the same time maintain high accuracy, because of the limitations on computational resources. Few approaches have been developed using analytical techniques that do not require sampling. However, they do not work for complex geometries. There are two main advantages of using analytical techniques. First, we can extract functions to determine cutter engagement at any point along the cutter path without

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explicitly resolving to expensive Boolean operations. Secondly, it is not based on sampling and hence it is more accurate than sampling techniques.

In this paper, we first describe analytical approaches to determine the cutter engagement using the engagement functions for individual half-spaces that comprise the workpiece geometry. Then, we introduce a set of rules to obtain the composition of these functions. After that, we describe a geometric algorithm to extract the cutter engagement functions based on analytical techniques for a given workpiece and a cutter path. The input to the algorithm is a two-dimensional boundary representation describing the workpiece geometry, tool diameter, and the cutter path. The main distinguishing feature of this algorithm is its ability to extract continuous closed-form functions that give the cutter engagement at every point along the cutter path. Finally, we have implemented discrete simulations of the cutting process and compare the results obtained by analytical techniques with that of discrete simulations.

2 RELATED RESEARCH

Kramer developed equations to calculate the engaged circumference of the cutting tool [Kramer 1992]. He presented an algorithm to detect minimal engagement for cuts along scan segments in the zigzag traversal while generating the cutter path itself, for rough pocket milling. The algorithm does not monitor the degree of tool engagement while machining along the pocket boundary. His algorithm is limited to determining minimal engagement along scan segments for zigzag pocket milling. Also, it cannot determine tool engagement along the pocket profile. In contrast, the algorithm presented in this paper can determine tool engagement for all 2.5D milling operations. It is not limited to any particular machining strategy, like zigzag. Also, it can determine tool engagement along the pocket boundary, with the limitation that the boundary should consist of linear and circular edges only.

Spence et al proposed an analytical method to calculate cutter engagement value [Spence et al. 1990]. They used filled circles and rectangles as primitives to describe parts to be machined, studied the cases when the cutter path was linear, and proposed two basic calculations, one was for the cutter engagement of circle primitives, one was for rectangles, and finally applied Boolean operations to find the entry and exit angles on the cutter. This approach works fine for cases where geometry is simple and can be constructed by these two types of primitives easily. However, in practice, parts may be more complex, and the cutter paths may include circular segments.

Stori and Wright developed a method to generate offset curves that would allow constant tool engagement [Stori and Wright 2000]. They presented an algorithm for 2.5D material removal of convex geometries. The algorithm continuously monitors the state of the machining process, including cutter engagement, path curvature and the instantaneous entry and exit angles. It generates a cutter path with constant given tool engagement. When constant engagement cannot be maintained, the path curvature entry and exit angles are controlled explicitly. A state based algorithm was presented to control the transition between the alternate strategies. It takes the tool engagement value as input and generates the cutter path so that the tool engagement remains constant. This is complimentary to our work presented in this paper, where the tool path is taken as the input and the tool engagement is the output. Moreover, our algorithm handles more complex geometric shapes.

Wang determined the tool engagement by subtracting tool swept volume from the workpiece [Wang 1988]. He transformed the problem of determining the tool swept volume to that of computing a family of curves, thus making it more efficient than the geometric Boolean of tool profile after every line of instruction in the cutter path [Wang 1984]. He integrated this information with a machinability database and a metal cutting model to determine the cutting force and dynamically update the feed rate so as to minimize the variation in the cutting force. Gu et al. computed the tool engagement as the range between entry and exit angles [Gu et al. 1997]. Entry and exit points were determined using the intersection of workpiece with the cutter. The in-process geometry of the workpiece was determined using the geometric Boolean operation. Tool swept volume for a cutter movement was subtracted from the workpiece to obtain the workpiece after that cut was machined.

3 ALGORITHM FOR COMPUTING CUTTER ENGAGEMENT

3.1 Definitions

In 2.5D milling operations, the depth of cut remains constant. Hence, the cutter engagement depends only on the profile of the face being machined and the trajectory of the cutter. The profile of the face can be defined using one or more outer edge loops and zero or more inner edge loops [Yao et al. 2003, Yao et al. 2001]. The loops must be none self-intersecting. This paper uses the convention of representing outer loops as counter-clockwise and inner loops as clockwise. This is to ensure that the material always lie on the left side of the loop boundary. Also, only linear and circular edges are allowed. The cutter path should also consist of linear and circular cutting moves only. In this paper, the milling cutter is modeled as a circle and the flutes are ignored. The following definitions will be required to build the theory in the subsequent sections.

- *Tool Swept Region*: This is the region covered by a moving cutter. Since we are working with 2.5D machining, only the 2D profile of the region is of much significance. The ends of the tool swept region (TSR) are semi-circular with diameter same as that of the cutter. The middle portion of TSR can have different shape depending on the cutter movement, as shown in Figure 2.

- *Tool Frontier*: A line perpendicular to the cutting direction and passing through the center of the cutter, divides the cutter's circumference into two parts; the one facing the cutting direction is defined as the tool frontier. It represents the portion of the cutter that can be actually involved in machining as shown in the Figure 2.

- *Half-Space*: A curve (infinite or closed) in 2D divides the plane into two parts. Each part is called a half-space. In this paper, the profile of the stock part and the machining geometry is composed of only linear and circular edges. Hence, only linear and circular half-spaces are of importance for this paper.

- *Solid Side*: The half-space that has the material to be removed.

- *Solid Angle*: When two solid-sides intersect, their union or intersection forms a solid angle. Whether the solid angle is obtained by union or intersection is determined by the angle value.

If the solid angle is greater than 180 degrees, it is obtained by the union of the two solid sides otherwise it is obtained from intersection.

- *Cutter Engagement*: A cutter may be engaged with several half-spaces simultaneously. Therefore, the cutter engagement for that location can be represented by a set of disjoint intervals on the frontier of the cutter. Each interval is represented by the difference of two angles as shown in Figure 3. Thus, finding these angles is important in determining cutter engagement. There are three different ways to measure cutter engagement: cutter engagement angle, cutter engagement point set, and cutter engagement fraction as shown in Figure 3. If not specified otherwise, in this paper, the cutter engagement refers to the cutter engagement fraction.

- *Cutter Engagement Function*: This is the function that takes the cutter location as input and returns the cutter engagement. In the case of intersection with a single half-space, this function is continuous and differentiable in its domain. But in case of intersection with multiple half-spaces, this function may not be differentiable at every point.

3.2 Computing Cutter Engagements with respect to Single Half-Spaces

In this paper, a cutter is represented as a circle. If (x_c, y_c) is the center of the cutter and r_c is its radius, any point on the boundary of the cutter is given by the equation:

$$(x-x_c)^2 + (y-y_c)^2 = r_c^2$$

The length of the cut is parameterized with respect to parameter t between 0 and 1. $t = 0$ corresponds to the start point and $t = 1$ corresponds to the end point of the cutter movement. Any other value of t gives an intermediate point on the cutter path segment. The cutter is not always engaged with the half-space. The value of parameter t, when it just starts to engage or disengage is called the *tangential parameter*. At that value of t, the cutter is tangential to the half-space. The value of parameter t when the cutter is just fully engaged or fully disengaged is called the *frontier parameter*. At this position, one of the end points of tool frontier coincides with the point of intersection of cutter with the half-space. If the cutter is moving into the solid side, the other end point of the tool frontier lies inside the solid side. Otherwise it lies outside the solid side. For linear cut, closed form expressions can be developed to determine t_s and t_f. But for circular cut, these values can only be determined using numerical techniques.

Each linear half-space is assigned a line. Any point on the line is given by the equation:

$$ax + by + c = 0$$

Each circular half-space is represented by a circle. If (x_h, y_h) is the center of the circular half-space and r_h is its radius, any point on its boundary is given by the equation:

$$(x-x_h)^2 + (y-y_h)^2 = r_h^2$$

Without losing generality, a linear cutter path segment can be represented as a line segment that starts from point (0, 0) and moves in $+x$ direction. When a cutter path segment is oriented differently, it can be reoriented along with the stock part without affecting the cutter engagement.

If the length of the cut is d, the position of the center of the cutter along the cutter path segment can be represented in parametric form as:

$$x_c = dt$$

$$y_c = 0$$

Using the same argument as above, a circular cutter path segment can be simplified by reorienting it such that it's center coincides with the origin and the start point is along $+x$ direction at $(R, 0)$, R being the radius of circular cut. If after transformation, the end point of the cut is at (x_e, y_e), the position of the center of the cut along the cutter path can be represented in parametric form as:

$$x_c = R\cos(kt)$$

$$y_c = R\sin(kt)$$

where,

$$k = \tan^{-1}\left(\frac{y_e}{x_e}\right)$$

Unlike the linear cut, the orientation of tool frontier changes along the circular cut. The end points of the tool frontier can be obtained by the intersection of line joining the cutter center with the origin and the circle representing the cutter.

Line joining origin with the cutter center, L: $x\sin(kt) - y\cos(kt) = 0$

Equation of cutter, C: $(x - R\cos(kt))^2 + (y - R\sin(kt))^2 = r_c^2$

End points of tool frontier:

$$F_1((R+r_c)\cos(kt), (R+r_c)\sin(kt))$$

$$F_2((R-r_c)\cos(kt), (R-r_c)\sin(kt))$$

If the cut is counter-clockwise, the tool frontier lies on the left of line L otherwise on the right. If any of the intersection points of the cutter with the half-space does not lie on the tool frontier, it is replaced by F_1 or F_2, whichever is closer.

Cutter engagement is non-zero in the following cases:

- Cutter is completely inside the solid side. In this case, the cutter does not intersect with the half-space boundary corresponding to the solid side and the cutter engagement is 100%. This case can be easily identified by examining the location of the cutter center with respect to the half-space.

- Cutter intersects the half-space boundary. The cutter engagement will vary from 0 to 100%.

It depends on the cutter trajectory and the type of half-space.

Our method for computing cutter engagement for linear segments is similar to the method described in [Spence et al. 1990]. The equations for the cutter as a parameterized function of time and the equation of the halfspace are simultaneously solved in order to yield the intersection points between the cutter and the halfspace as a function of time. Using this, along with the path position partitioning presented by [Spence et al. 1990], the entry and exit angles can be calculated and used to calculate cutter engagement. The cutter path segments used in [Spence et al. 1990] are linear only. In many applications, circular cuts are often used. Although one can assume the circular cutter path can be decomposed into a set of smaller linear segments, this would be more computationally expensive as well as result in a loss in accuracy. In our approach, we not only can handle linear cutter path, but also the circular one. Our approach studied four basic cases, i.e., a) linear cutter path with respect to linear half space; b) linear cutter path with respect to circular half space; c) circular cutter path with respect to linear half space; and d) circular cutter path with respect to circular half space. In [Spence et al. 1990], the authors proposed two basic calculations, which are similar to our first two cases, however, they did not study the other two cases, which will be described in detail in this paper in Appendix A.

3.3 Computing Cutter Engagement with respect to Multiple Half-Spaces

Cutter path and stock geometry are rarely so simple that cutter engages with a single edge (corresponding half-space) in a single move. Hence, it is needed to determine cutter engagement with respect to multiple half-spaces. A Boolean expression that relates multiple half-spaces can represent any region consisting of multiple different edges [Peterson 1986, Shapiro and Vossler 1991]. This is illustrated in Figure 4. The region can be expressed as ($a.b + c + d + e.f$). The Boolean expression can be used to compose the engagement function for the multiple half-spaces using the cutter engagement functions for individual half-spaces.

The tool swept region may divide the given region into several *connected-regions.* A connected-region is the sequence of connected edges. The engagement function is composed for each connected-region separately, independent of other connected-regions.

Figure 5 shows the cutter engagement functions for the half-spaces and the cutter path shown in Figure 4. As can be seen from it, the cutter engagement starts and ends at different values of t for every half-space. To compose the overall engagement function, the entire *t-interval* is divided into sub-intervals. The splitting is done at the place where the cutter engagement is 0% or 100% for every half-space. The t-interval is further divided where the engagement functions intersect for edges within a connected-region. After splitting the t-interval, the composition is done for each sub-interval separately.

The individual engagement functions for each half-space can be combined in exactly the same way as the half-spaces are combined together to form the connected-region. This is known from the Boolean expression for the connected-region. Thus, if the two half-spaces are combined using Union, their engagement functions are also combined using Union. The way in which functions are combined is now explained. Here, α is the composite engagement function and α_1 and α_2 are the individual engagement functions.

$$\alpha_1 = ((\theta_{11} - \theta_{12}) / \pi) \times 100$$

$$\alpha_2 = ((\theta_{21} - \theta_{22}) / \pi) \times 100$$

- *Union*: It refers to the adding of engagement functions. There are two possible cases:
  - Disjoint Intervals: As shown in Figure 6, the two engagement intervals do not overlap. In this case, the composite engagement function is just the sum of the individual engagement functions.

    $$\alpha = \alpha_1 + \alpha_2$$

  - Overlapping Intervals: As shown in Figure 7, the two engagement intervals overlap. In this case, the composite engagement function is given by the *outer* bounds of the individual engagement functions. Thus, in Figure 7(a),

    $$\alpha = ((\theta_{11} - \theta_{22}) / \pi) \times 100$$

    and in Figure 7(b),

    $$\alpha = ((\theta_{11} - \theta_{12}) / \pi) \times 100$$

- *Intersection*: This kind of composition refers to the situation when the presence of one half-space limits the engagement interval of the other half-space. There are two different cases:
  - Disjoint Intervals: As shown in Figure 6, the two engagement intervals do not overlap. In this case, the composite engagement is simply zero.

    $$\alpha = 0$$

  - Overlapping Intervals: As shown in Figure 7, the two engagement intervals overlap. In this case, the composite engagement function is given by the *inner* bounds of the individual engagement functions. Thus, in Figure 7(a),

    $$\alpha = ((\theta_{21} - \theta_{12}) / \pi) \times 100$$

    and in Figure 7(b),

    $$\alpha = ((\theta_{21} - \theta_{22}) / \pi) \times 100$$

The following proposition provides the basis for constructing an algorithm for combining engagement resulting from different half spaces.

Proposition: Let S be the point set representing a 2D connected-region and E the cutter engagement point set of a cutter with respect to S. Let C be the set of points representing the tool frontier. Suppose $S = \boldsymbol{B}\ (H_i,\ i = 1, \ldots, n)$, where H_I's are half-spaces that form S using Boolean operator $\boldsymbol{B}$ (the Boolean operators used in this paper are union $\cup$, intersection $\cap$ and difference -) and for each half-space H_i, the engagement point set for a cutter is given by $E_i = C \cap H_i$. Then the cutter engagement point set can be obtained by $E = \boldsymbol{B}\,(E_i,\ i = 1, ..., n)$.

Proof: By using Venn diagram, it can be shown that given sets X, Y, and Z, the following relationships hold: $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$, $X \cap (Y \cap Z) = (X \cap Y) \cap (X \cap Z)$, and $X \cap (Y - Z) = (X \cap Y) - (X \cap Z)$ (variations of distributive law [Suppes 1972]). We know $E_i = C \cap H_i$ and $E = C \cap S$. As $S = \boldsymbol{B}\,(H_i, i = 1, ..., n)$, therefore, $E = C \cap \boldsymbol{B}\,(H_i, i = 1, ..., n) = \boldsymbol{B}\,(C \cap H_i, i = 1, ..., n) = \boldsymbol{B}\,(E_i, i = 1, ..., n)$.

Thus, if the Boolean expression for a connected-region is known in form of linearly ordered sequence, the composite cutter engagement function can be computed by combining engagement functions for two half-spaces at a time starting from the left. The result of the first composition is then combined with the next engagement function using the corresponding Boolean operator. This process is continued until all the half-spaces for a connected region in that sub-interval are processed.

The same procedure is carried out for all the sub-intervals and for all the connected-regions. Once the composite engagement functions for each connected-region are determined, they are combined using the Union operator within each sub-interval to obtain the final composite cutter engagement function for that particular cutter path segment.

3.4 Computing Cutter Engagement Functions for the Part Program

In this section, an algorithm is described to compute the cutter engagement functions for a particular cutter path segment.

Procedure: CALCULATE_TSR_ENGAGEMENT (W, Q, P, TSR, r)

Input: Current stock geometry W, Set of affected edges Q, Cutter path segment P, tool swept region TSR, Cutter radius r.

Output: Set of composite engagement functions for the whole cutter path segment.

Steps:

1. Compute the intersection of tool swept region with the 2D region and determine different connected-regions.
2. For each connected-region c_i:
    a. For each edge segment e_i:
        i. Assign half-space along e_i as h
        ii. Fc_i = CALCULATE _INDIVIDUAL_ENGAGEMENT (h, P, r)
    b. Determine the Boolean expression that represents the connected-region c_i
    c. Compose the engagement function for the connected-region using engagement

functions of constituent half-spaces

3. Compose the overall engagement function for all connected-regions into α

4. Return α

Procedure: CALCULATE _INDIVIDUAL_ENGAGEMENT (*h*, *p*, *r*)

Input: Half space *h*, Cutter path segment *p*, Cutter radius *r*.

Output: Tool engagement function, i.e., a composition of functions describing $\theta_1(t)$ and $\theta_2(t)$, in the range $t = 0$ to 1.

Steps:

1. Get the equation of the circle corresponding to cutter, as a function of tool center coordinates, which are represented in parametric form.

2. Compute the intersection points by simultaneously solving the circle and half space equations.

3. Calculate the angles of intersection, $\theta_1(t)$ and $\theta_2(t)$

4. Compose α containing $\theta_1(t)$ and $\theta_2(t)$

5. Return α

3.5 Time Complexity of the Algorithm

The procedure CALCULATE_TSR_ENGAGEMENT has the worst case time complexity of $O(E^2)$, where E is the number of edges of the workpiece part at a given instant of time. This can be shown in the following way:

Step 1 computes the intersection of tool swept region with the stock part and determines the different connected regions. This requires every edge of the tool swept region should be checked for intersection with every edge of the stock part. For only linear and circular cutter movements, the number of edges on the tool swept region is either 3 or 4. Hence, this step can be finished in $O(E)$ time.

Step 2 computes the engagement function for all connected regions. Step 2.a requires splitting of *t-interval* into various sub-intervals. The individual engagement function for any half-space corresponding to an edge on the stock part can intersect with any other individual cutter engagement function. If there are e_i edges in the i^{th} connected region, then the total number of sub-intervals within that connected region varies as $O(e_i)^2$. If there are m connected regions and E number of edges on the stock part, then the following condition holds:

$$\sum_{i=1}^{m} e_i = E$$

The complexity of the Step 2 is $O\left(\sum_{i=1}^{m} e_i^2\right)$.

Since $\sum_{i=1}^{m} e_i^2 \le \left(\sum_{i=1}^{m} e_i\right)^2 = E^2$

Therefore, the complexity of Step 2.a is $O(E^2)$. Because the worst case complexity of Step 2.b and Step 2.c are also $O(E^2)$, the complexity of Step 2 is $O(E^2)$.

Step 3 composes the engagement function for all the connected regions. Since the number of connected regions is $O(E)$, this step also runs in $O(E)$ time. Finally, Step 4 runs in constant time.

Hence the overall complexity of procedure CALCULATE_TSR_ENGAGEMENT is also $O(E^2)$.

The complexity of algorithm COMPUTE_ENGAGEMENT_FUNCTIONS is determined by the number of cuts (N) and the complexity of the intermediate workpiece geometry. Suppose E_m is the maximum number of edges encountered in Step 4, then the complexity of the algorithm is $O(NE_m^2)$.

4 RESULTS

We implemented the algorithm described in Section 3 using Matlab. Figure 8a-b show a representative stock and workpiece on which our algorithm was tested. The stock was the starting geometry and the workpiece was the final geometry. The workpiece was constructed from a combination of circular and straight edges and the part program (shown in Figure 8c) was a combination of straight and circular cuts. Therefore this test case covered all the combinations of half spaces and cutter segment types. The tool size used was 1 inch.

The cutter engagement was computed and represented as a function of distance the cutter has traveled for the entire part program. Figure 9 shows a plot of the complete cutter engagement function. The graph is broken into segments to correlate with the different segments of the part program shown in the figure.

In order to validate the results of the analytical technique, we were interested in comparing the results obtained by the analytical techniques with the discrete simulation of the machining process. Therefore, we also implemented a system for performing discrete simulation of the machining process. The basic technique behind discrete simulation of cutting process is based on using a 2D regular grid of cells and marking the cells based on their status and computing cutter engagement at discrete time interval. The cutter engagement in percentage is given by *Cutter Engagement = 100(n/N).* Where, n = number of un-machined cells getting machined at the current cutter position; and N = total number of cells below the cutter frontier.

The stock and part program shown in Figure 8 were used by the discrete simulation program for determining cutter engagement and the results of this program were compared to the analytical results. In the simulation, cell size of 0.005 inch and tool step of 0.05 inch were used. The plot of the numerical solution is plotted simultaneously with the analytic solution in Figure 9. It can be seen that the results of the two algorithms agree to within 2%.

5 CONCLUSIONS

This paper describes an analytical approach to computing cutter engagement functions. The edges of workpiece are represented as half-spaces. This paper presents formulae to determine the cutter engagement functions with the half-spaces in the following cases: circular cut and linear half-space, and circular cut and circular half-space. It also describes a mathematically sound approach to compose the overall cutter engagement function from the individual engagement functions. After that, it presents an algorithm to compute the cutter engagement function for a given cutter path segment and presents the worst-case asymptotic time complexity analysis for this algorithm. We validated the results obtained from the analytical technique by comparing those results to results obtained from discrete simulation of machining process. We found the two results to be in agreement.

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APPENDIX A

A.1 Circular Cut and Linear Half-Space

When the cutter is moving along a circular cut and intersects a linear half-space, the intersection points P_1 (x_1, y_1) and P_2 (x_2, y_2) can be determined by the simultaneous solution of the equation of the cutter and the equation of the line. The basic equations are the same as that for linear cut and linear half-space case [Spence et al. 1990], with the following differences:

- The position of the center of the cutter is determined by trigonometric functions instead of simple linear functions in case of linear cut.

- The orientation of tool frontier varies along the cut, i.e., it depends on the value of t.

We use the following equation of a line:

$ax + by + c = 0$

There are two possible cases:

Case 1: $a = 0$, i.e. horizontal half-space, then

$$y_1 = y_2 = -\frac{c}{b}$$

$$x_1 = R\cos(kt) + \sqrt{r_c^2 - \left(\frac{c}{b} + R\sin(kt)\right)^2},\ x_2 = R\cos(kt) - \sqrt{r_c^2 - \left(\frac{c}{b} + R\sin(kt)\right)^2}$$

Case 2: $a \neq 0$, i.e. any arbitrary orientation of linear half-space, then

$$y_1 = \frac{-B + \sqrt{B^2 - AC}}{A} = \frac{-B + \sqrt{D}}{A},\ y_2 = \frac{-B - \sqrt{B^2 - AC}}{A} = \frac{-B - \sqrt{D}}{A}$$

$$x_1 = -\frac{b}{a}y_1 - \frac{c}{a},\ x_2 = -\frac{b}{a}y_2 - \frac{c}{a}$$

where,

$$A = a^2 + b^2$$
$$B = b_1 \cos(kt) + b_2 \sin(kt) + b_3$$
$$C = c_1 \cos(kt) + c_2$$
$$D = d_1 \cos^2(kt) + d_2 \sin^2(kt) + d_3 \cos(kt)\sin(kt) + d_4 \cos(kt) + d_5 \sin(kt) + d_6$$

$b_1 = abR$; $b_2 = -a^2R$; $b_3 = bc$;
$c_1 = 2acR$; $c_2 = a^2(R^2-r_c^2) + c^2$;
$d_1 = b_1^2$; $d_2 = b_2^2$; $d_3 = 2b_1b_2$;
$d_4 = 2b_1b_3 - c_1(a^2+b^2)$; $d_5 = 2b_2b_3$; $d_6 = b_3^2 - c_2(a^2+b^2)$

Once the intersection points are obtained, the corresponding values of entry and exit angles can be calculated as:

$$\theta_1 = \tan^{-1}\left(\frac{y_1 - R\sin(kt)}{x_1 - R\cos(kt)}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{y_2 - R\sin(kt)}{x_2 - R\cos(kt)}\right)$$

This is true for all circular cuts.

Figures A1-A3 show an example of a circular cut into a linear halfspace as well as plots of engagement angle versus time and cutter engagement versus time for this case.

A.2 Circular Cut and Circular Half-Space

When a cutter moving along a circular path intersects a circular half-space, the intersection points P_1 (x_1, y_1) and P_2 (x_2, y_2) can be determined by the simultaneous solution of the equation of the cutter and the equation of the circle. The two circles intersect if and only if the following condition holds:

$$(r_h - r_c)^2 < (x_h - R\cos(kt))^2 + (y_h - R\sin(kt))^2 < (r_h + r_c)^2$$

Equation of half-space: $(x - x_h)^2 + (y - y_h)^2 = r_h^2$

Equation of cutter: $(x - R\cos(kt))^2 + (y - R\sin(kt))^2 = r_c^2$

There are two possible cases:

Case 1: $R\sin(kt) - y_h = 0$, then

$$x_1 = x_2 = \frac{N}{2A} + x_h$$

$$y_1 = y_h + \sqrt{r_h^2 - \left(\frac{r_c^2 - A^2 - r_h^2}{2A}\right)^2},$$

$$y_2 = y_h - \sqrt{r_h^2 - \left(\frac{r_c^2 - A^2 - r_h^2}{2A}\right)^2}$$

where,

$$N = r_c^2 - A^2 - r_h^2$$
$$A = x_h - R\cos(kt)$$
$$B = y_h - R\sin(kt)$$

Case 2: $R\sin(kt) - y_h \neq 0$, then

$$[\,x_1' \quad x_2'\,] = [\,\frac{-K + \sqrt{K^2 - 4JL}}{2J} \qquad \frac{-K - \sqrt{K^2 - 4JL}}{2J}\,]$$

$$[\,x_1 \quad x_2\,] = [\,x_1' + x_h \qquad x_2' + x_h]$$

$$[\,y_1' \quad y_2'\,] = [\,\frac{Q}{2B} - \frac{A}{B}x_1' \qquad \frac{Q}{2B} - \frac{A}{B}x_2'\,]$$

$$[\, y_1 \quad y_2 \,] = [\, y_1' + y_h \qquad y_2' + y_h]$$

where,

$$J = 4A^2 + 4B^2$$

$$K = 4A^3 - 4Ar_c^{\,2} + 4Ar_h^{\,2} + 4AB^2$$

$$L = -2A^2r_c^{\,2} + 2A^2r_h^{\,2} + 2A^2B^2 - 2r_c^{\,2}r_h^{\,2} - 2r_c^{\,2}B^2 + 2r_h^{\,2}B^2 + A^4 + B^4 + r_c^{\,4} + r_h^{\,4}$$

$$A = x_h - R\cos(kt) \qquad B = y_h - R\sin(kt)$$

$$Q = r_c^{\,2} - r_h^{\,2} - A^2 - B^2$$

Figures A4-A6 show an example of a circular cut into a circular halfspace as well as plots of engagement angle versus time and cutter engagement versus time for this case.

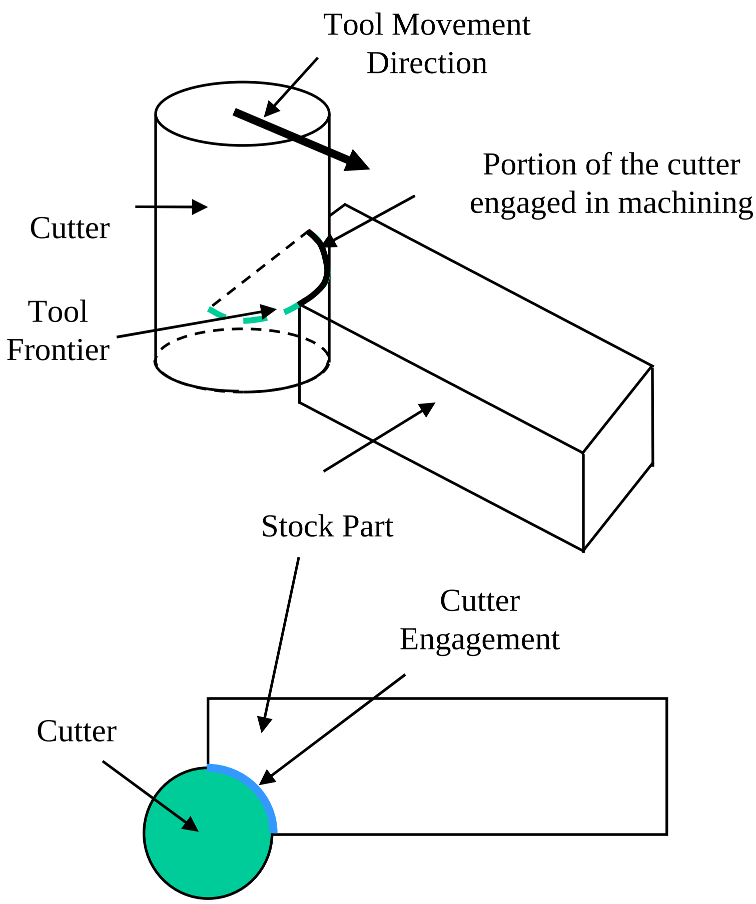


Figure 1: Cutter Engagement in Milling Operation

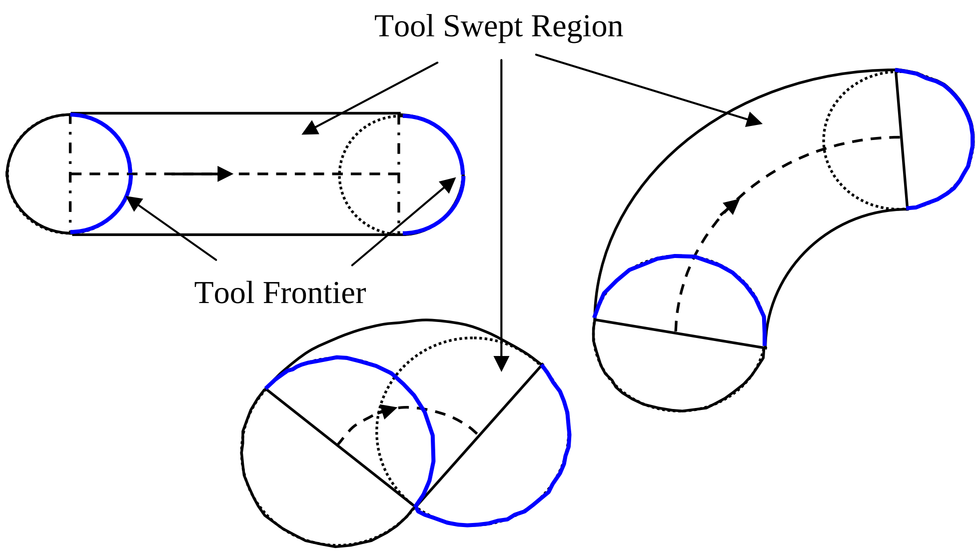
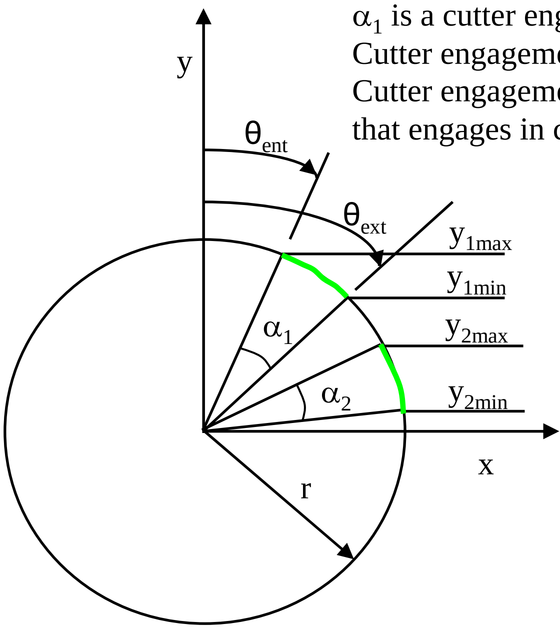


Figure 2: Tool Swept Region and Tool Frontier

α_1 is a cutter engagement angle.
Cutter engagement fraction is a percentage value (= 100 α_1/180).
Cutter engagement point set is the set of points on cutter frontier that engages in cutting.



$$\alpha_1 = \theta_{ent1} - \theta_{ext2}$$

$$\theta_{ent1} = \frac{\pi}{2} - \sin^{-1}\left(\frac{y_{1max}}{r}\right)$$

$$\theta_{ext2} = \frac{\pi}{2} - \sin^{-1}\left(\frac{y_{1min}}{r}\right)$$

Figure 3: Illustration of Cutter Engagement

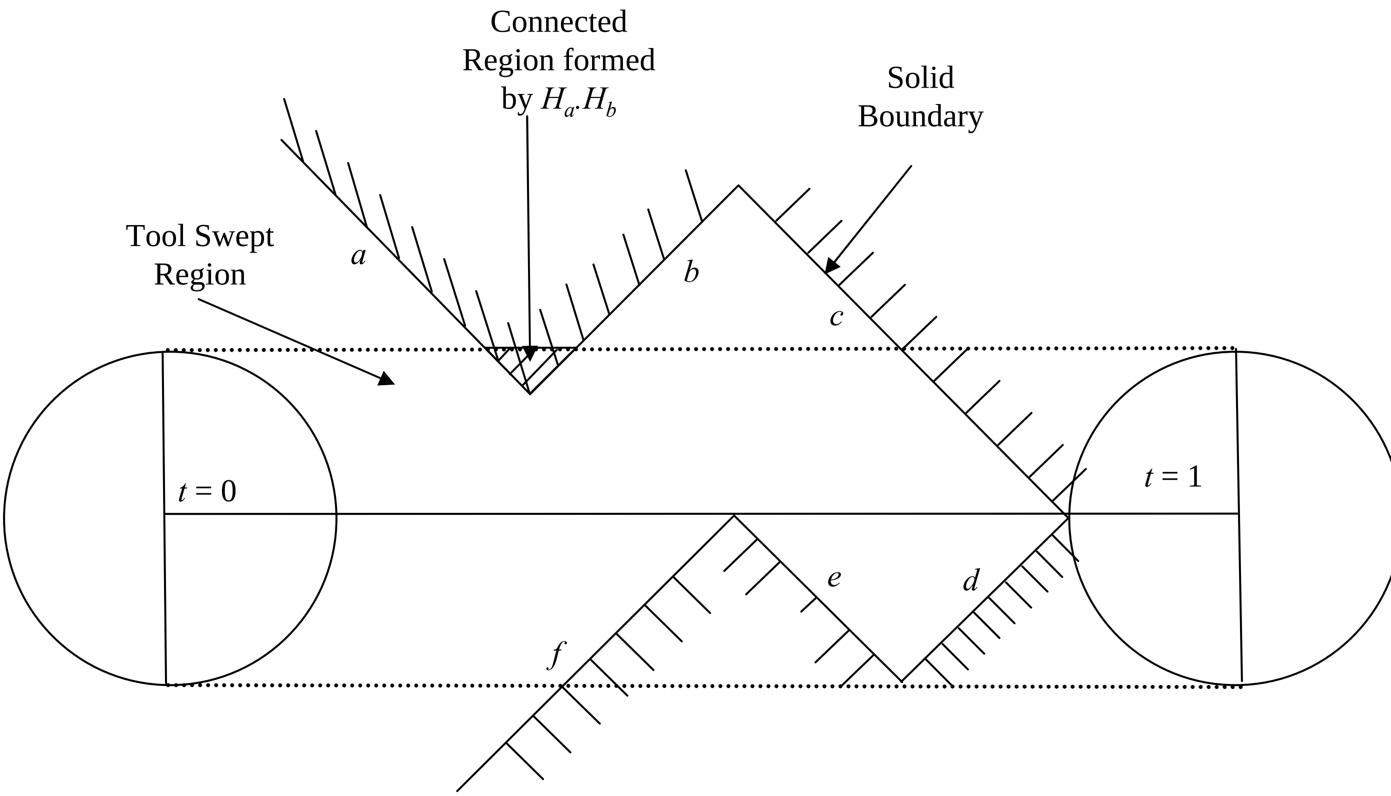


Figure 4: A Linear Cutter Path and A Region Expressed as a Boolean Expression of Half Spaces

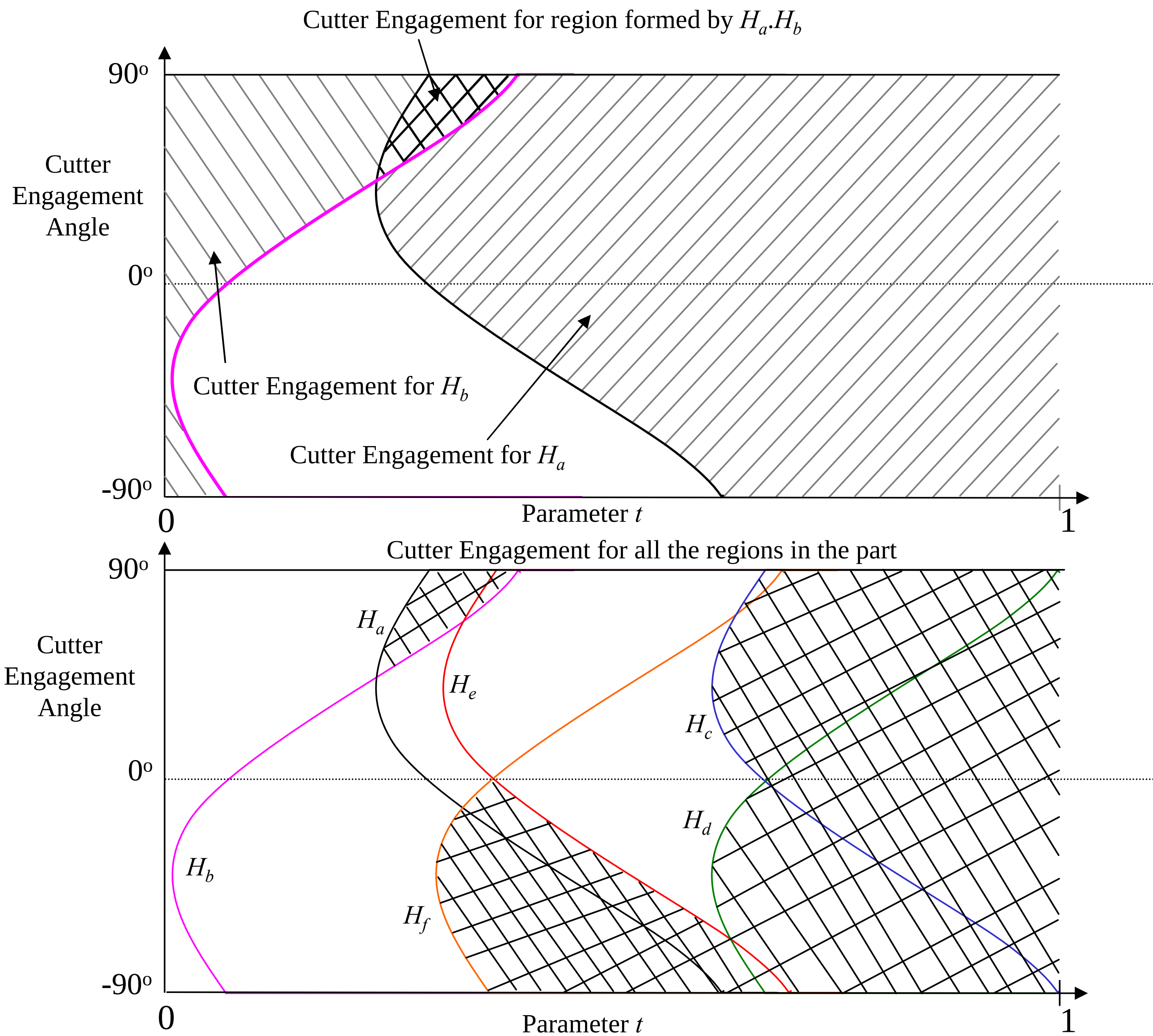


Figure 5: Cutter Engagement for Half-Spaces shown in Figure 4

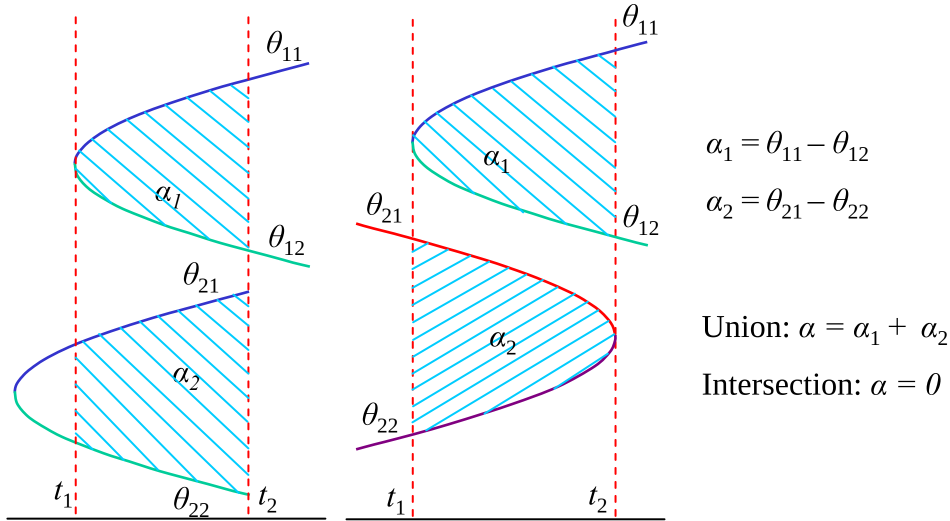
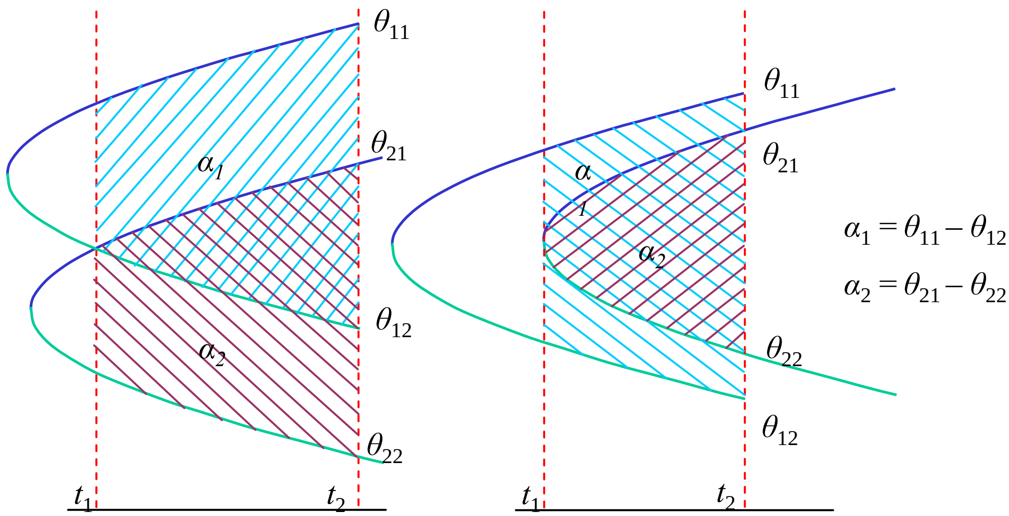


Figure 6: Composition of Engagement Functions for Disjoint Intervals



Union: $\alpha = \theta_{11} - \theta_{22}$

Intersection: $\alpha = \theta_{21} - \theta_{12}$

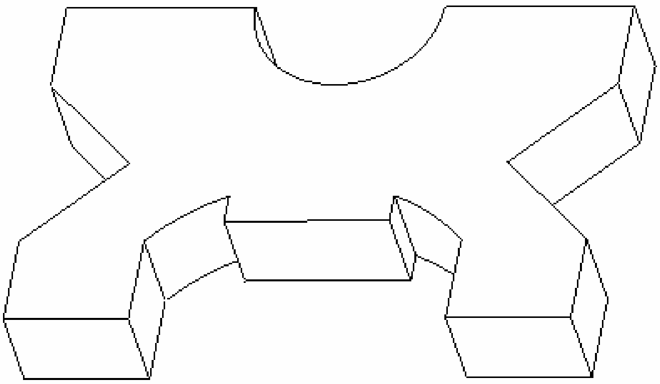
(a): Intersecting Interval

Union: $\alpha = \theta_{11} - \theta_{12}$

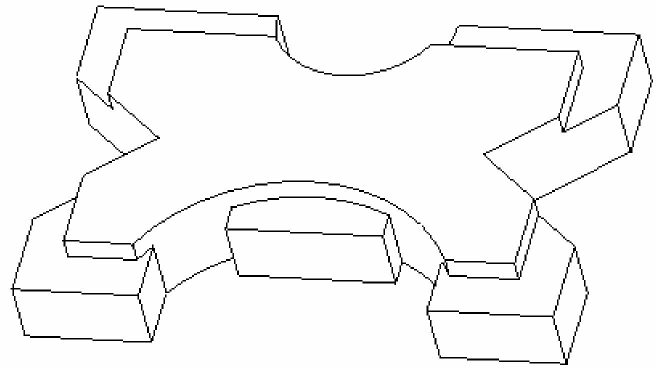
Intersection: $\alpha = \theta_{21} - \theta_{22}$

(b): Embedded Interval

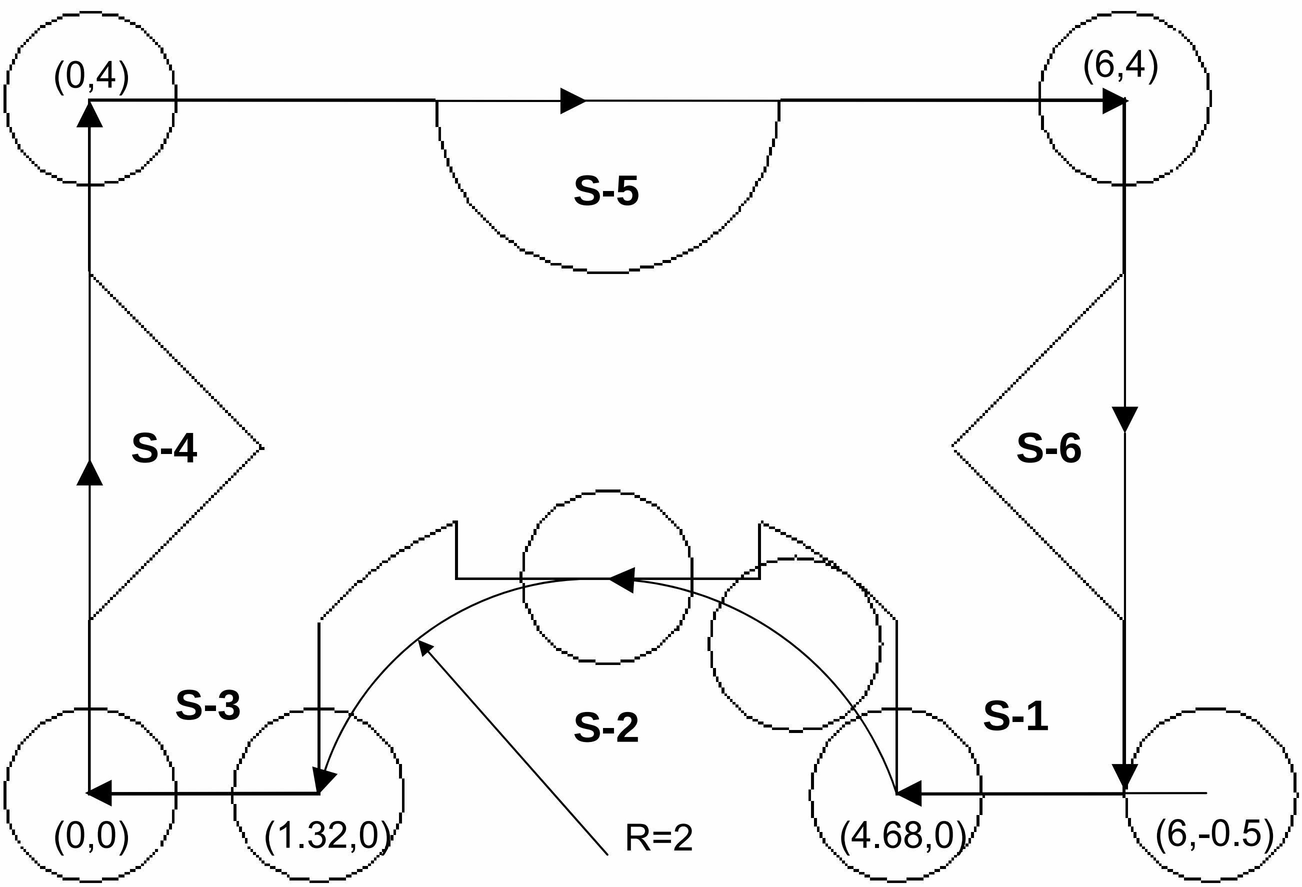
Figure 7: Composition of Engagement Functions for Overlapping Intervals



a) Stock



b) Workpiece



c) Cutter Path

Figure 8: Example of a Cutter Engagement Calculation

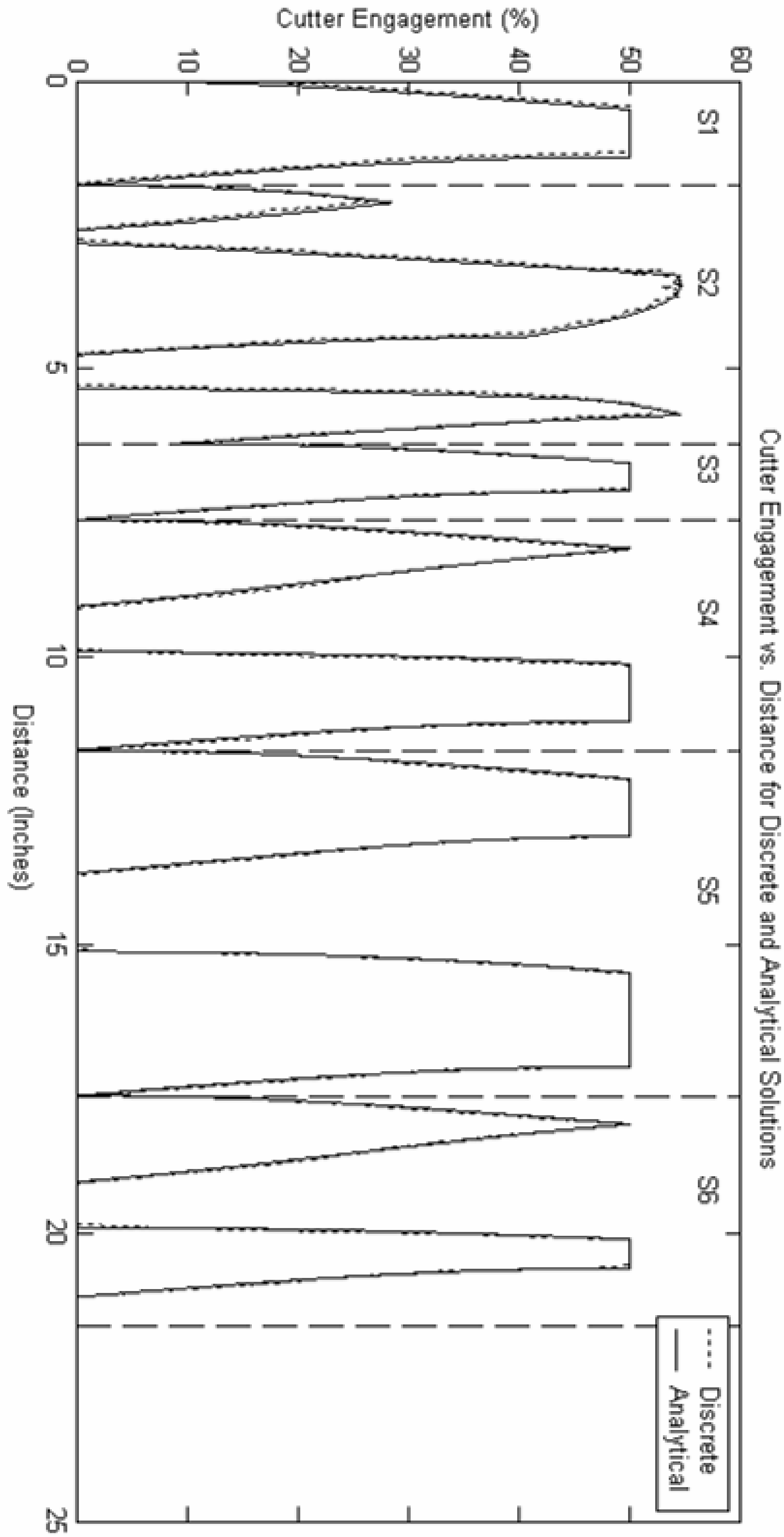


Figure 9: Cutter Engagement vs. Distance for Sample Operation

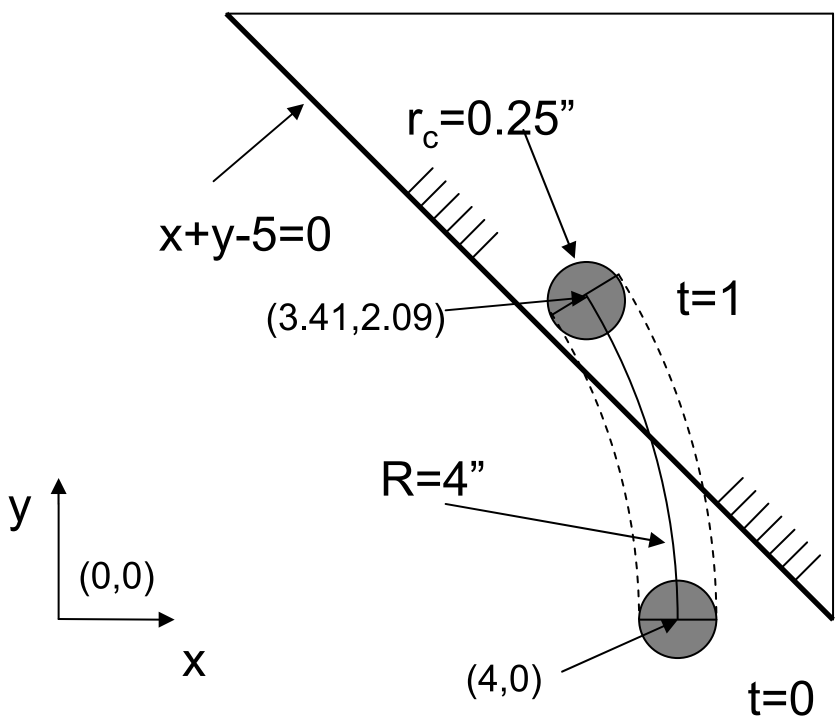


Figure A1: Circular Cut into Linear Half Space

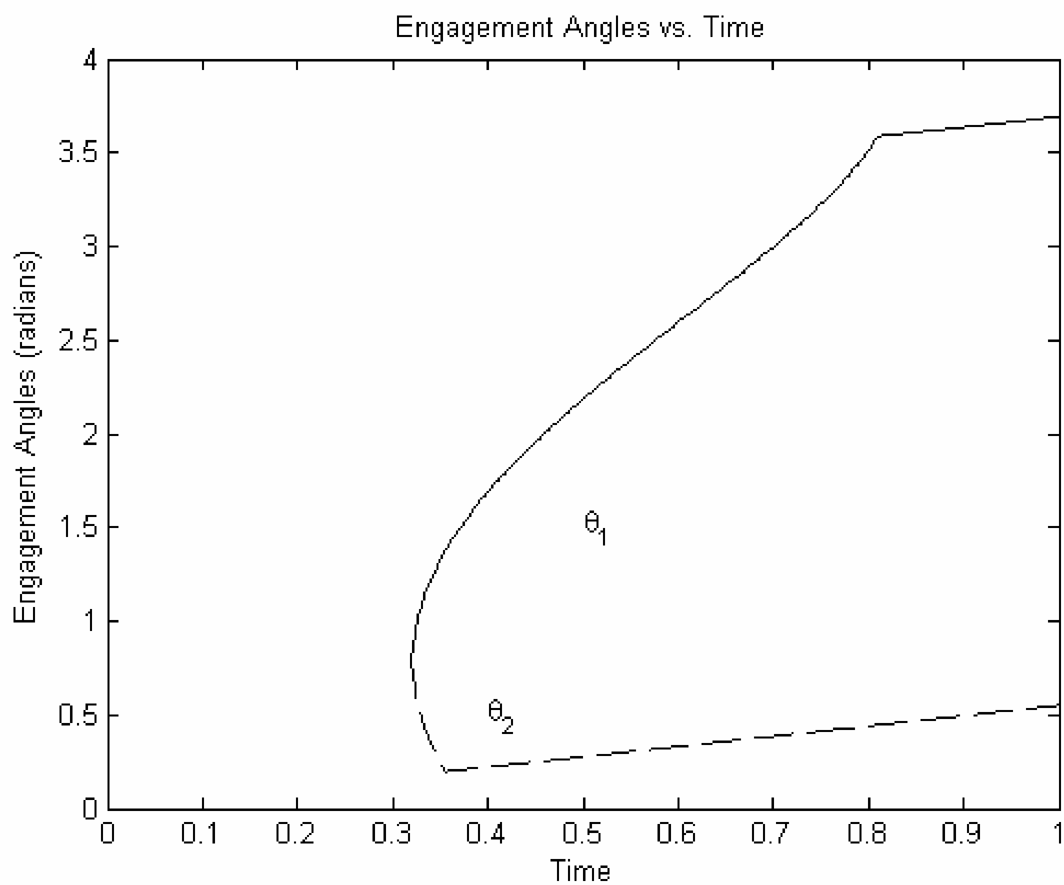


Figure A2: Engagement Angles vs. Time for Circular Cut and Linear Half Space Case

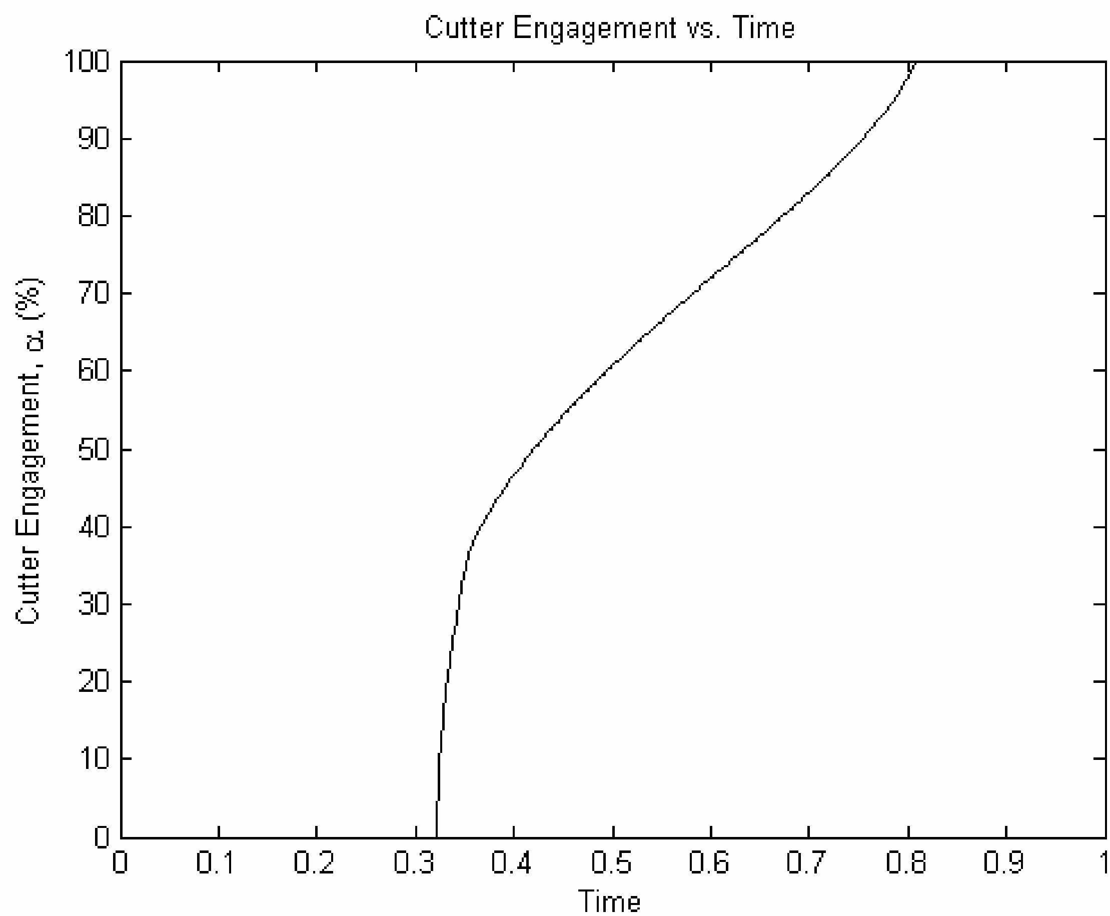


Figure A3: Cutter Engagement vs. Time for Circular Cut and Linear Half Space Case

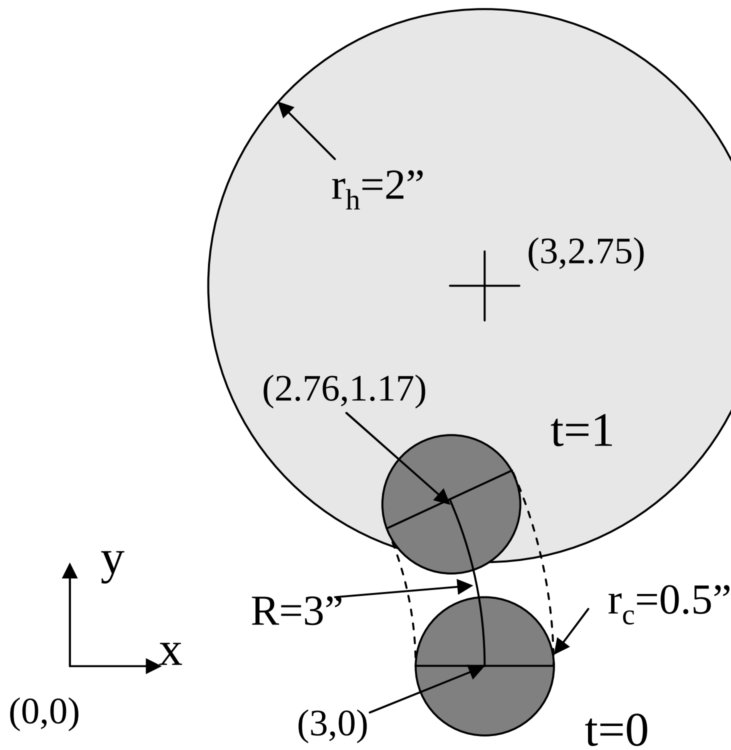


Figure A4: Circular Cut into Circular Half Space

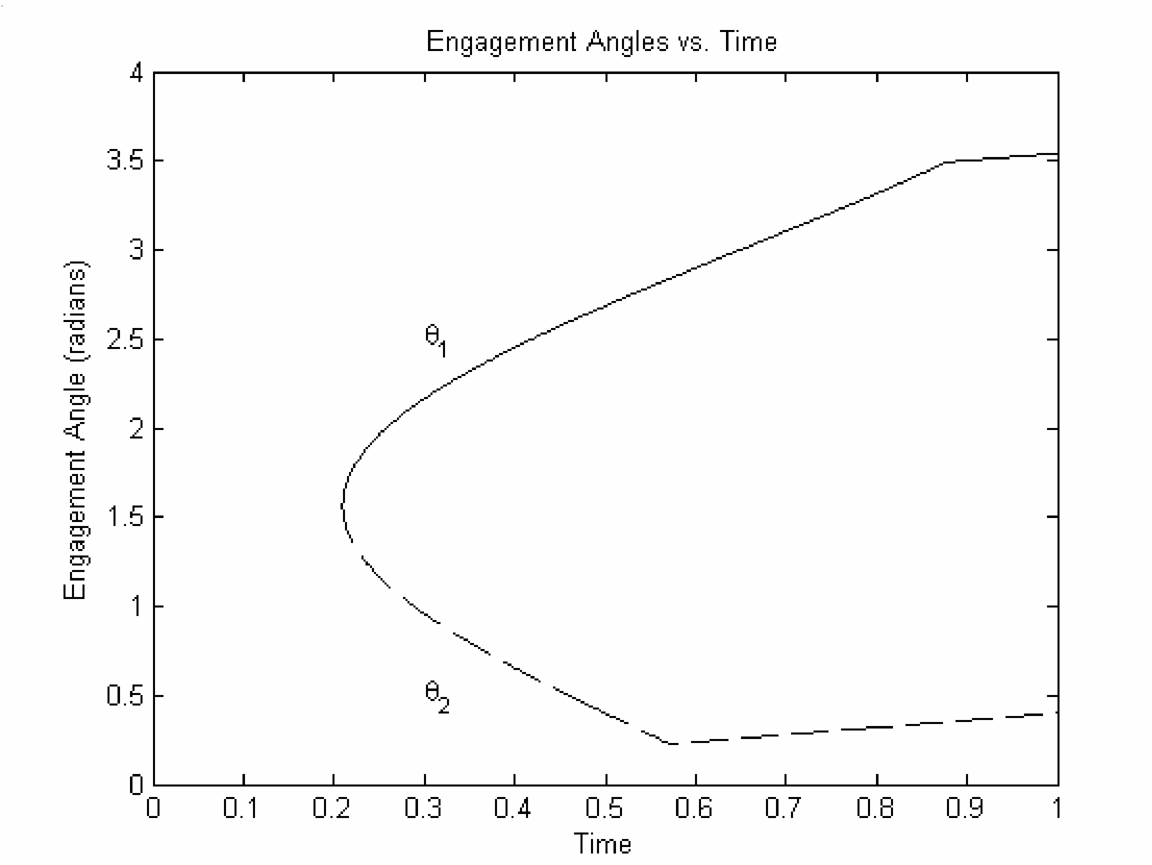


Figure A5: Engagement Angles vs. Time for Circular Cut and Circular Half Space Case

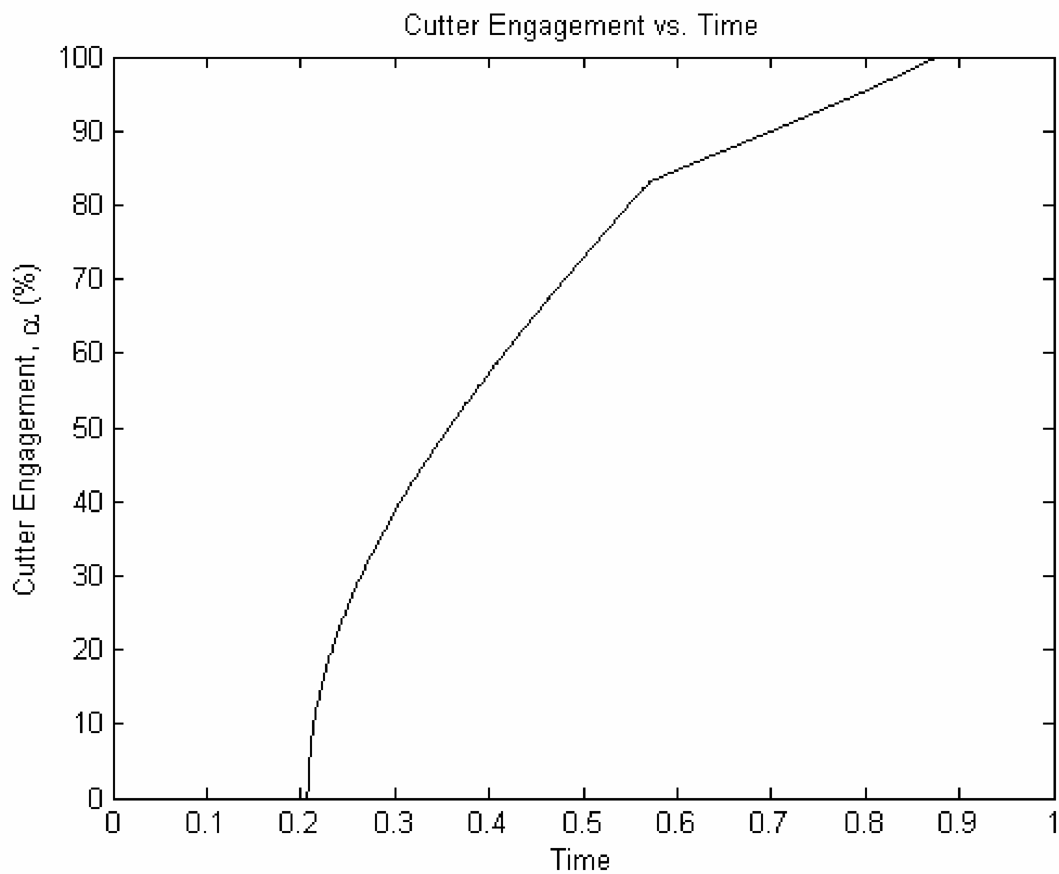


Figure A6: Cutter Engagement vs. Time for Circular Cut and Circular Half Space Case