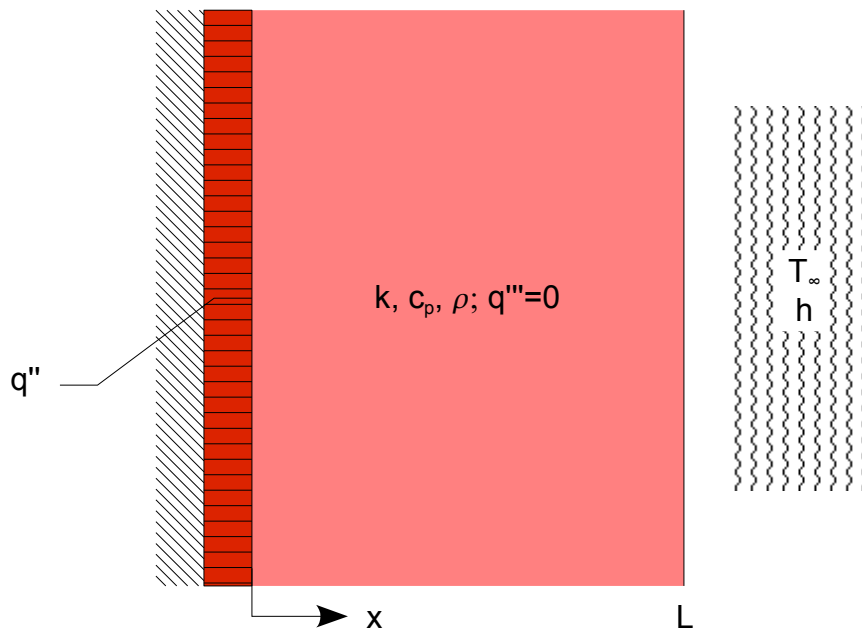


2.47 A plane wall has constant properties, no internal heat generation, and is initially at a uniform temperature T_i . Suddenly, the surface at $x = L$ is heated by a fluid at T_∞ having a convection coefficient h . At the same instant, the electrical heater is energized, providing a constant heat flux q_0'' at $x = 0$.



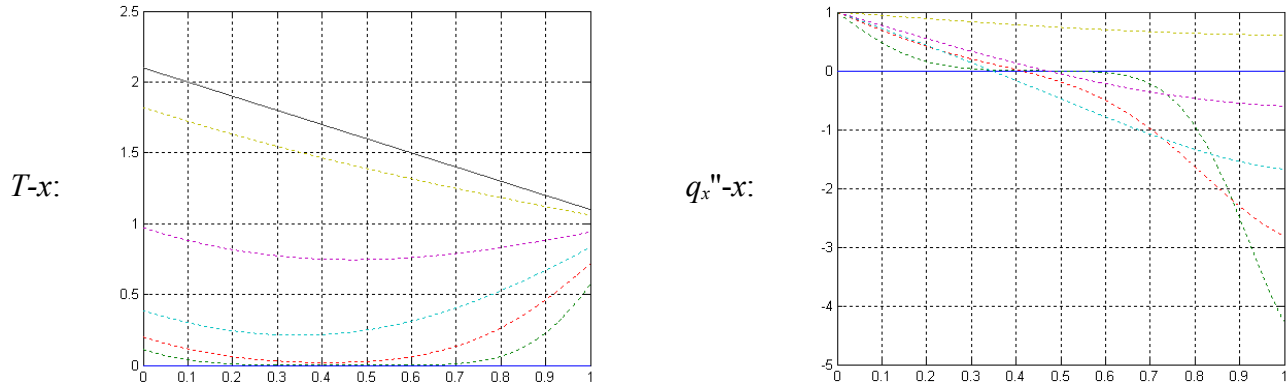
- (a) On T - x coordinates, sketch the temperature distributions for the following conditions: initial condition ($t \leq 0$), steady-state condition ($t \rightarrow \infty$), and for two intermediate times.
- (b) On q_x'' - x coordinates, sketch the heat flux corresponding to the four temperature distributions of part (a).
- (c) On q_x'' - t coordinates, sketch the heat flux at the locations $x = 0$ and $x = L$. That is, show qualitatively how $q_x''(0,t)$ and $q_x''(L,t)$ vary with time.
- (d) Derive an expression for the steady-state temperature at the heater surface, $T(0,\infty)$, in terms of q_0'' , T_∞ , k , h , and L .

Notes:

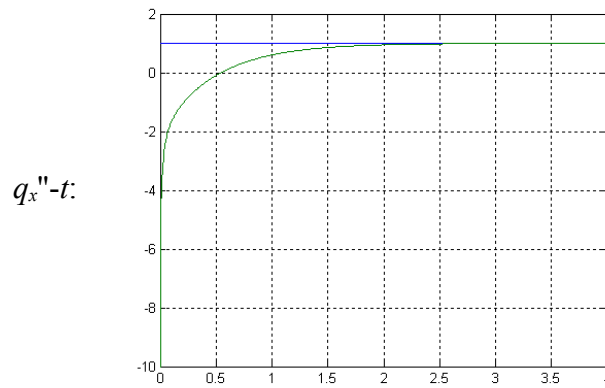
- Attached 'sketches' are MATLAB plots based on the exact solution, with parameters $k = 1$, $h = 10$, $L = 1$, $q_0'' = 1$, $T_\infty = 1$; $\alpha = k/\rho c_p = 1$; $T_i = 0$.
- The solution of (d) may be incorrect in the solution manual. Please check your work.

2.47 Solution

(a,b) Spatial plots in time order: blue ($t = 0$), green, red, cyan, purple, yellow, black (steady-state)



(c) Time plots: blue, $x = 0$; green, $x = L$



(d) At steady-state, as the properties and cross-section are constant and there is no internal heat generation or dissipation, we can conclude that the temperature profile is linear. Further, we may employ the discrete form of the conduction equation, as q_x'' is a constant throughout the slab.

Thus:

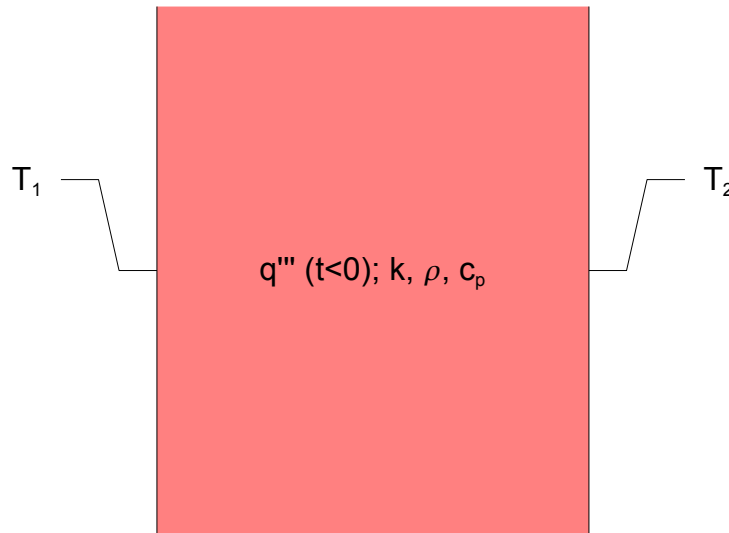
$$q_x'' = q_0'' = k \frac{T(0, \infty) - T(L, \infty)}{L} = h(T(L, \infty) - T_\infty)$$

$$T(L, \infty) = T_\infty + q_0'' / h$$

$$T(0, \infty) = T_\infty + (L/k) q_0''$$

$$T(0, \infty) = T_\infty + (1/h + L/k) q_0''$$

Example R1* Consider a plane wall with uniform properties, nonzero internal heat generation q''' , and fixed boundary temperatures T_1 and T_2 . (We may imagine a phase-change heat transfer process is occurring at each boundary such that $h \gg k/L$.) At times $t < 0$, the system is assumed to be in thermal equilibrium such that $\max(T(x, 0)) > T_1 > T_2$. At time $t = 0$, the internal heat generation is shut off.



- (a) Determine expressions for the temperature and heat flux distributions for the initial condition ($t < 0$) and the final steady-state condition ($t \rightarrow \infty$) in terms of the q''' , k , L , T_1 , and T_2 .
- (b) On T - x coordinates, sketch the temperature distribution for the following conditions: the initial condition, the steady-state condition, and two intermediate conditions.
- (c) On q_x'' - t coordinates, sketch the variation with time of the heat flux at each boundary.
- (d) Calculate the net amount of energy removed from the wall per unit area (J/m^2) as the wall cools from its initial to its steady-state condition in terms of ρ , c_p , q''' , L , and k .

* Example written to illustrate concept - does not appear in book.

'Sketches' calculated with $k = 1, L = 1, T_1 = 1, T_2 = 0, q''' = 4, \alpha = k/\rho c_p = 1$.

(a) Beginning with the constant-properties heat conduction equation and the boundary conditions:

$$\frac{\partial^2 T}{\partial x^2} + \frac{q'''}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} = 0 \quad (\text{steady-state})$$

$t=0$

$$T_0(x) = (-q'''/2k)x^2 + C_1x + C_2$$

$$T_0(0) = C_2 = T_1$$

$$T_0(L) = (-q'''/2k)L^2 + C_1L + C_2 = T_2$$

$$C_1 = (T_2 - T_1)/L + q'''L/2k$$

$$T_0(x) = (-q'''/2k)x^2 + [(T_2 - T_1)/L + q'''L/2k]x + T_1$$

$t \rightarrow \infty$

$$T_{ss}(x) = C_3x + C_4$$

$$T_{ss}(0) = C_4 = T_1$$

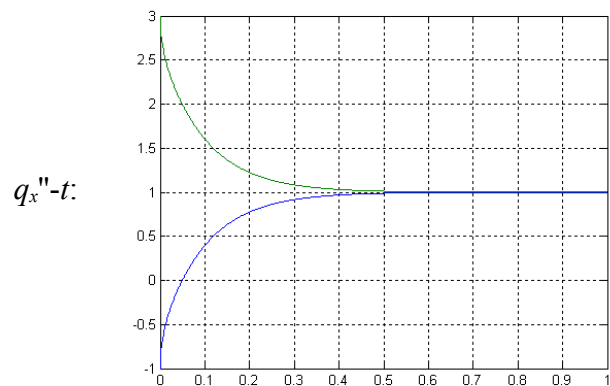
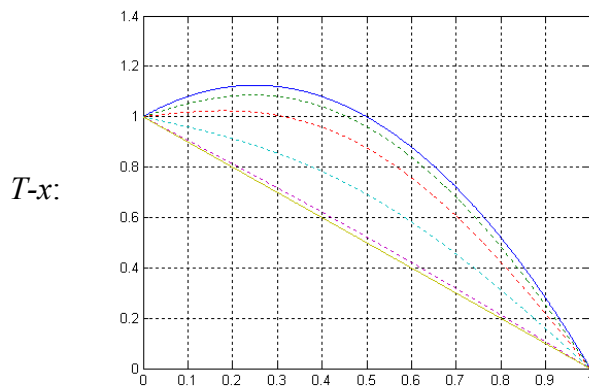
$$T_{ss}(L) = C_3L + C_4 = T_2$$

$$C_3 = (T_2 - T_1)/L$$

$$T_{ss}(x) = [(T_2 - T_1)/L]x + T_1$$

(b,c) Spatial plot: in time order: blue ($t = 0$), green, red, cyan, purple, yellow (steady-state)

Time plot: blue, $x = 0$; green, $x = L$



(d) As there is no energy generated within the wall during the time period in question, the net amount of energy removed must equal the net change in the energy of the wall. In algebraic form, from the conservation of energy, we can write:

$$\begin{aligned} Q''_{out} &= -Q''_{net} \\ \Delta E'' &= Q''_{net} - W''_{net} = Q''_{net} \\ Q''_{out} &= -\Delta E'' \end{aligned}$$

As E is a function of T , this latter quantity may be calculated as follows:

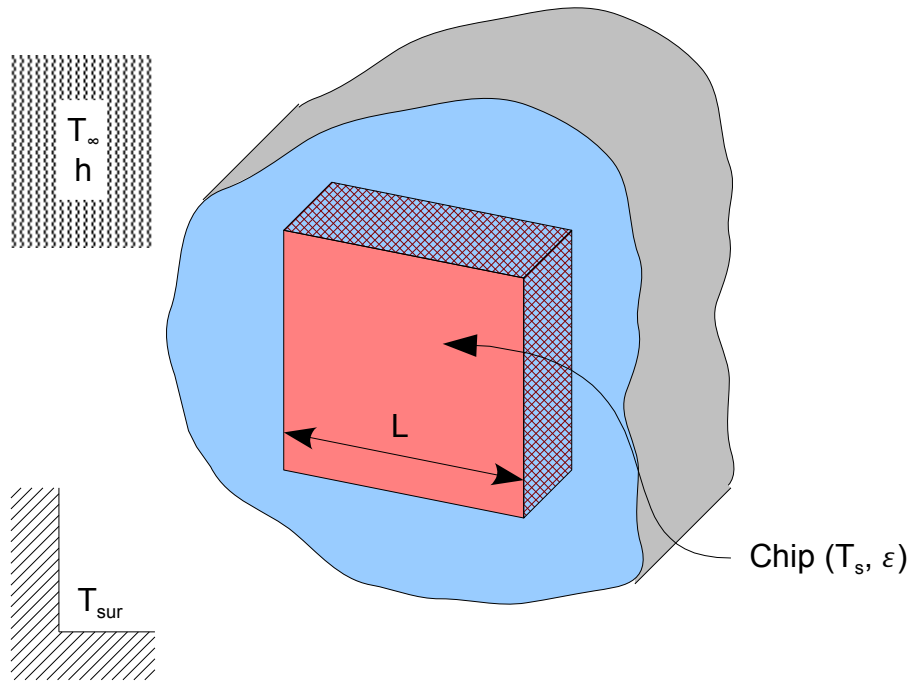
$$\Delta E = \Delta U = m c_p \Delta T$$

$$\begin{aligned} \Delta E'' &= \int_0^L \Delta E''' dx = \int_0^L \rho c_p \Delta T dx \\ &= \int_0^L \rho c_p (T_{ss} - T_0) dx \end{aligned}$$

Substituting the expressions for T_{ss} and T_0 into this expression, and this expression into that for Q''_{out} :

$$\begin{aligned} Q''_{out} &= \int_0^L \rho c_p \{ (-q'''/2k)x^2 + [(T_2 - T_1)/L + q'''L/2k]x + T_1 - [(T_2 - T_1)/L]x - T_1 \} dx \\ &= \int_0^L \rho c_p [(-q'''/2k)x^2 + (q'''L/2k)x] dx \\ &= (\rho c_p q'''/2k) \int_0^L (-x^2 + Lx) dx \\ &= (\rho c_p q'''/2k) [-x^3/3 + Lx^2/2]_0^L \\ Q''_{out} &= \frac{\rho c_p q''' L^3}{12k} \end{aligned}$$

1.31 Chips of width $L = 15 \text{ mm}$ on a side are mounted to a substrate that is installed in an enclosure whose walls and air are maintained at a temperature of $T_{sur} = T_{\infty} = 25 \text{ }^{\circ}\text{C}$. The chips have an emissivity of $\epsilon = 0.60$ and a maximum allowable temperature of $T_s = 85 \text{ }^{\circ}\text{C}$.



(a) If heat is rejected from the chips by radiation and natural convection, what is the maximum operating power of each chip? The convection coefficient depends on the chip-to-air temperature difference and may be approximated as $h = C(T_s - T_{\infty})^{(1/4)}$, where $C = 4.2 \text{ W/m}^2\text{-K}^{5/4}$.

(b) If a fan is used to maintain air flow through the enclosure and heat transfer is by forced convection with $h = 250 \text{ W/m}^2\text{-K}$, what is the maximum operating power?

Bonus Question: Imagine the chips of (b) are running when the cooling fan shuts off. Assuming the power consumption remains constant and the free convection formula from (a) holds, what temperature will the chips reach?

Notes:

- The chips are assumed to be thin (thickness t such that $k/t \gg h$), and therefore of uniform temperature.
- The substrate is assumed to be perfectly insulating.

(a) As the substrate is insulating and the system is at the steady-state, we can determine the power consumption by an energy balance:

$$\text{operating power} \equiv q = q'' A = \text{heat dissipated}$$

Using the given relations for heat flux and substituting the relation for h :

$$\begin{aligned} q'' &= C(T_s - T_\infty)^{1/4}(T_s - T_\infty) + \epsilon \sigma (T_s^4 - T_{sur}^4) \\ &= 4.2(85 - 25)^{5/4} + 0.60 \cdot 5.67 \left[\left(\frac{85 + 273}{100} \right)^4 - \left(\frac{25 + 273}{100} \right)^4 \right] \\ &= 991.9 \text{ W/m}^2 \end{aligned}$$

$$\begin{aligned} q &= q'' A = (991.9 \text{ W/m}^2)(0.015 \text{ m})^2 \\ \mathbf{q} &= \mathbf{0.223 \text{ W}} \end{aligned}$$

Note that Celsius temperatures are corrected to Kelvins for radiation, and the 10^{-8} of the Stefan-Boltzmann constant is taken inside the fourth power to reduce the range of numbers. Note also the value of h at this temperature:

$$h = 4.2(85 - 25)^{1/4} = 11.69 \text{ W/m}^2 \cdot \text{K}$$

(b) The procedure is identical save for the value of h :

$$\begin{aligned} q'' &= h(T_s - T_\infty) + \epsilon \sigma (T_s^4 - T_{sur}^4) \\ &= 250(85 - 25) + 0.60 \cdot 5.67 \left[\left(\frac{85 + 273}{100} \right)^4 - \left(\frac{25 + 273}{100} \right)^4 \right] \\ &= 15291 \text{ W/m}^2 \end{aligned}$$

$$\begin{aligned} q &= q'' A = (15291 \text{ W/m}^2)(0.015 \text{ m})^2 \\ \mathbf{q} &= \mathbf{3.44 \text{ W}} \end{aligned}$$

Bonus: The conservation principle is the same, but now the unknown variable is T_s :

$$\begin{aligned} q'' &= C(T_s - T_\infty)^{1/4}(T_s - T_\infty) + \epsilon \sigma (T_s^4 - T_{sur}^4) \\ 15291 &= 4.2(T_s - 25)^{5/4} + 0.60 \cdot 5.67 \left[\left(\frac{T_s + 273}{100} \right)^4 - \left(\frac{25 + 273}{100} \right)^4 \right] \end{aligned}$$

We note, however, that this is a nonlinear equation. How are we to solve for T_s ? There are many methods; I shall list but a few.

(Oh, so as not to keep you in suspense: $T_s = 424.6$ °C. At this temperature, $h = 18.8$ W/m²-K; 50.9% of the power (7787 W/m²) is dissipated by radiation. This state is *far* out of the operating specifications of the system - the system might well catch fire before reaching this temperature. Many real systems have thermal sensors to prevent this sort of thing.)

'Obvious' methods:

- Graphing
 - Advantages: Easy.
 - Disadvantages: Imprecise, slow, requires graphing calculator or computer.
- Analytical solution (e.g. "quartic equation" - see Wikipedia article)
 - Advantages: Exact, guaranteed to find all roots.
 - Disadvantages: Impossible for many problems, slow, may require calculator or computer to obtain decimal solutions, may require identifying the correct root among multiple roots.
- Automated numerical solution (e.g. nSolve() on TI-89/92 series calculators, fsolve() in MATLAB)
 - Advantages: Easy, fast, accurate, reliable.
 - Disadvantages: Not guaranteed to give correct root, hard to debug in case of problems ("black-box"), requires computer or sophisticated graphing calculator.

Manual numerical methods:

- Bisection method
 - Advantages: Straightforward, robust.
 - Disadvantages: Very slow.
- Newton's method
 - Advantages: Very fast.
 - Disadvantages: Requires differentiation, may fail to converge or converge to incorrect root.
- Secant method
 - Advantages: Fast.
 - Disadvantages: May fail to converge or converge to incorrect root.
- Regula falsi (false position) method
 - Advantages: Robust.
 - Disadvantages: Irregular convergence speed (up to as fast as the secant method).

Note 1: the automatic numerical solvers you find today are good enough that it is not worth the effort to solve equations manually unless you lack a sufficiently powerful calculator or computer. (Plus, I'm not actually an authority on numerical methods, unlike - hopefully - the programmers of the automatic routines, so I can't tell you with any authority which method is best.)

Note 2: I mention the bisection method mainly for completeness. Under almost all circumstances, you would choose one of the other, faster methods, if only to avoid making a mathematical error among all the calculations you has to make. If you can't make anything else work, however, the bisection method is unlikely to let you down.

Note 3: All of these manual methods require acquiring (through whatever means) estimates of the solution that are at least marginally accurate. In most cases, physical intuition based on figures given in the problem statement will suffice for this purpose in these problems. ($T_s = 85$, $T_s = 100$, and $T_s = 1000$ are all good enough for this problem, for example.)

Note 4: All of these methods require that the function be continuous in the vicinity of the root. This will generally be true for the functions of this class.

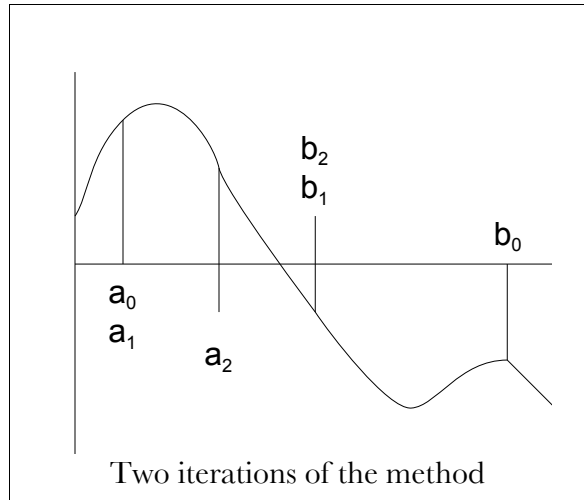
Bisection Method:

1. Write problem in the form $f(x) = 0$, where x is the variable to be solved for.
2. Find two points a_0 and b_0 near the root such that $f(a_0)$ and $f(b_0)$ are of opposite sign. Say that $f(a_0)$ is positive and $f(b_0)$ is negative.
3. Calculate

$$c_0 = \frac{a_0 + b_0}{2}$$

and $f(c_0)$.

4. If $f(c_0)$ is negative, set $a_1 = a_0$ and $b_1 = c_0$. If $f(c_0)$ is positive, set $a_1 = c_0$ and $b_1 = b_0$. This will leave you with two points like those you began with, save that the interval containing the root is half as long.
5. Repeat 3 and 4 until your interval (i.e. error) is small enough for your satisfaction (e.g. $|a - b| < |c|/2000|$ for three significant figures). Note that you can always predict how long this will take.



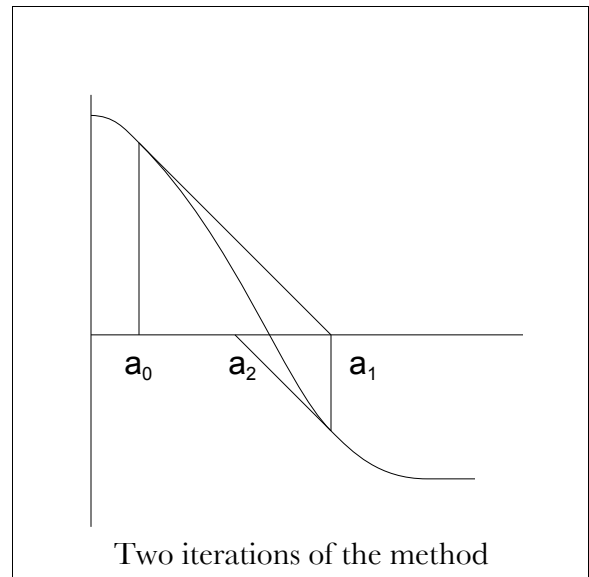
When to avoid this method: When other methods can be safely used; when the root being sought is likely to have even multiplicity (e.g. $x = 0$ for $f(x) = x^2$; $f(x) > 0$ on both sides of $x = 0$).

Newton's Method:

Also known as the Newton-Raphson or Newton-Fourier method.

1. Write problem in the form $f(x) = 0$, where x is the variable to be solved for.
2. Calculate the derivative function $f'(x)$.
3. Find a point a_0 near the root.
4. Calculate $f(a_0)$ and $f'(a_0)$.
5. Calculate

$$a_1 = a_0 - f(a_0) / f'(a_0)$$
 (see figure for graphical interpretation).
6. Repeat 4 and 5 until the corrections between repetitions is small enough for your satisfaction (e.g. $|f(a_n) / f'(a_n)| < |a_n|/2000|$).



When to avoid this method: When the function has large derivatives and a good initial guess is not available; when the function derivative is difficult to obtain; when the root being sought is likely to have multiplicity higher than one.

Secant Method:

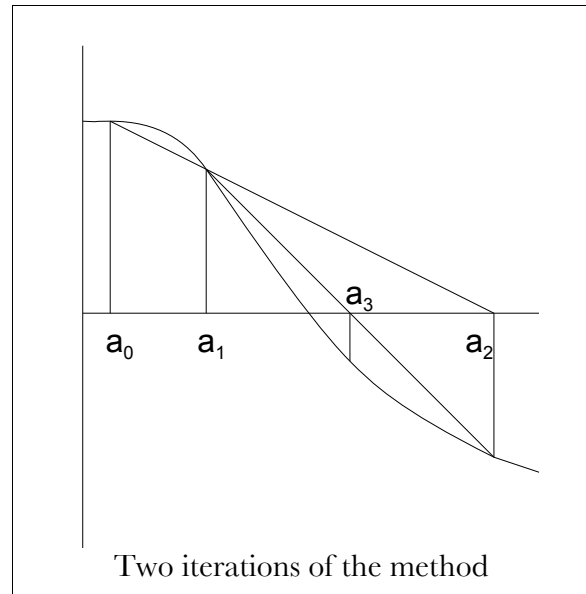
1. Write the problem in the form $f(x) = 0$, where x is the variable to be solved for.
2. Find two distinct points a_0 and a_1 near the root.
3. Calculate $f(a_0)$ and $f(a_1)$.
4. Calculate

$$a_2 = a_1 - \frac{a_1 - a_0}{f(a_1) - f(a_0)} f(a_1)$$

$$= \frac{f(a_1)a_0 - f(a_0)a_1}{f(a_1) - f(a_0)}$$

(see figure for graphical interpretation - notice that the second term in the first formula is a discrete approximation of $f(a)/f'(a)$).

5. Repeat 3 and 4 with the two latest a -values until the corrections between repetitions is small enough for your satisfaction.



When to avoid this method: When the function has large derivatives and a good initial pair of guesses is not available; when Newton's method is easier.

Regula Falsi (False Position) Method:

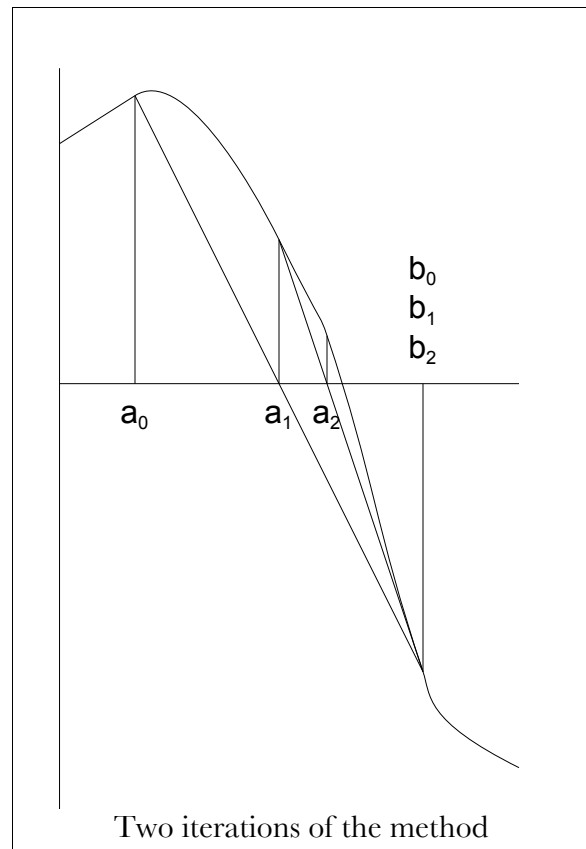
1. Write the problem in the form $f(x) = 0$, where x is the variable to be solved for.
2. Find two points a_0 and b_0 near the root such that $f(a_0)$ and $f(b_0)$ are of opposite sign. Say that $f(a_0)$ is positive and $f(b_0)$ is negative.
3. Calculate

$$c_0 = \frac{f(a_0)b_0 - f(b_0)a_0}{f(a_0) - f(b_0)}$$

(the secant-method formula).

4. If $f(c_0)$ is negative, set $a_1 = a_0$ and $b_1 = c_0$. If $f(c_0)$ is positive, set $a_1 = c_0$ and $b_1 = b_0$ (the bisection method procedure). This will give you two points like those you began with, save that the interval containing the root is shorter.
5. Repeat 3 and 4 with the latest pair of a, b values until the corrections between repetitions of a or b is small enough for your satisfaction.

Note. Sometimes using the Regula Falsi method will result in one point converging slowly to the solution while the other remains unchanged for many repetitions. One way to correct this is to



alter the formula whenever the same endpoint is retained twice in a row by calculating

$$c = \frac{w f(a)b - f(b)a}{w f(a) - f(b)} \quad \text{or} \quad c = \frac{f(a)b - w f(b)a}{f(a) - w f(b)}$$

where $0 < w < 1$ is a factor on the value at the point that was retained twice. This will force the point in question to be replaced for the next iteration if w is sufficiently small.

When to avoid this method: When the root being sought is likely to have even multiplicity; when Newton's method or the secant method is reliable.

For purposes of this class, if you need to use a manual method, I would recommend Newton's method or the secant method, and resort to another method only if those do not work. Starting at $T_s = 85$, Newton's method returned an answer correct to five significant figures in five iterations, and to the limits of the TI-89 (~14 sig. figs.) in seven repetitions.