Practice problems for Exam 2

1. We are given the following information about $f(x)$:

$$f(0) = 2, \quad f(1) = 1, \quad f(3) = 0, \quad f(4) = 1$$

(a) Write down the divided difference table. Find interpolating polynomial in Newton form (i) for the nodes in the order $0, 1, 3, 4$, (ii) for the nodes in the order $4, 3, 1, 0$.

(b) Assume we know that the 4th derivative satisfies $\left|f^{(4)}(x)\right| \leq 10$ for $x \in [0, 4]$. Find an upper bound for $|f(2) - p(2)|$.

2. Consider the (x, y) data points $(-1, 2)$, $(1, 1)$, $(2, 0)$. We want to fit the data with a function $g(x) = c_1 + c_2 x^2$

(a) Find the best least squares fit by hand.

(b) Write a Matlab program which uses `qr` to solve this problem.

3. We want to find $c \in \mathbb{R}^2$ such that $\|Ac - y\|_2$ is minimal. Here $y = [1, 2, 3, 0]^\top$, and for the matrix A we have the QR decomposition $A = QR$ with

$$Q = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}.$$

Use this to find the solution vector c. DO **NOT** COMPUTE $A = QR$. DO **NOT** USE THE NORMAL EQUATIONS FOR THE MATRIX A.

4. We want to find x such that $x + x^5 = 3$.

(a) Perform one step of the bisection method with $a_0 = 1$, $b_0 = 2$. Find k such that $|b_k - a_k| \leq 10^{-6}$.

(b) Perform one step of the secant method with $x_0 = 1$, $x_1 = 2$ to find x_2.

(c) Will the Newton method converge if we start with x_0 sufficiently close to the solution x_*? Explain.

5. Consider the nonlinear system

$$x_1 + x_1 x_2 + x_2 = 2, \qquad x_1 - x_2 - x_1 x_2^2 = 0$$

(a) Perform one step of the Newton method starting with initial guess $x^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) Write a Matlab program which uses the Newton method to find a solution, starting with initial guess $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The program should print out the approximation for $\mathbf{x}$ after each iteration.

6. Consider the nonlinear system

$$x_1 = \frac{1 + x_2 + \cos(x_1 + x_2)}{4}$$
$$x_2 = \frac{1 + x_1 + \sin(x_1 - x_2)}{4}$$

(a) Show that this nonlinear system has a unique solution in the square $[0, 3] \times [0, 3]$. *Hint:* Use the ∞-norm.

(b) Starting with $x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ find $x^{(1)}$ using the fixed point iteration. What can you conclude about the location of the solution x^*? Give the answer in the form $x^* \in [a_1, b_1] \times [a_2, b_2]$.
Hint: Use the a-posteriori estimate $\left\|x^{(k)} - x^*\right\| \leq \frac{q}{1-q}\left\|x^{(k)} - x^{(k-1)}\right\|$.