Assignment #3, due Wednesday, May 9

1. We consider the heat equation $u_t - 2u_{xx} = 0$ on the interval $[0,1]$ with boundary conditions

$$u(0,t) = 0, \qquad u_x(1,t) = 0$$

and initial condition $u(x,0) = 1$. In the series for the solution $u(x,t)$ find the **first two terms**. *Hint:* $\int_0^{k\pi/2} \sin^2(z)dz = \frac{k\pi}{4}$ for integer k.

2. We consider the wave equation $u_{tt} - 4u_{xx} = 0$ on the interval $[0,1]$ with boundary conditions

$$u(0,t) = 0, \qquad u'(1,t) = 0$$

and initial conditions $u(x,0) = u_0(x) = 0$, $u_t(x,0) = u_1(x) = 1$.

(a) Use the appropriate extension to define the function $\tilde{u}_1(x)$ for all $x \in \mathbb{R}$ and sketch the graph of this function. *Hint:* use an even extension at Neumann boundary, odd extension at Dirichlet boundary.

(b) Write down the D'Alembert formula for the extended solution $\tilde{u}(x,t)$. Use this to find $u(\frac{1}{2}, \frac{1}{2})$: mark the interval over which you have to integrate $\tilde{u}_1(x)$ on your graph of $\tilde{u}_1(x)$; then find the value of $u(\frac{1}{2}, \frac{1}{2})$. Evaluate $u(x, \frac{1}{2})$ for $x \in [0,1]$.

3. Consider a square metal plate $G = [0,1] \times [0,1]$. At three sides it is cooled to temparature 0 (Dirichlet condition), at the remaining side it is insulated (Neumann condition). The temperature $u(x,y,t)$ satisfies the heat equation $u_t - 2\Delta u = 0$. We start with the initial temperature $u_0(x,y) = 1$. At what rate λ will the temperature decay, i.e., $|u(x,y,t)| \le ce^{-\lambda t}$? For large t give an approximation to $u(x,y,t)$. *Hint:* Find a solution of the form $e^{-\lambda t}v(x,y)$ with the smallest possible λ and find the coefficient C so that $u(x,y,t) = Ce^{-\lambda t}v(x,y) +$ faster decaying terms.

4. Consider a square membrane $G = [0,1] \times [0,1]$ which is fixed at three sides (Dirichlet conditions) and free at the remaining side (Neumann conditions). The displacement $u(x,y,t)$ satisfies the wave equation $u_{tt} - 4\Delta u = 0$. What is the lowest frequency ω which the membrane can generate? *Hint:* Find a solution of the form $u(x,y,t) = \cos(\omega t)v(x,y)$ with the smallest possible ω.