1. Consider the integral \( I = \int_0^2 x^3 \, dx \).

(a) Find the value of the (i) midpoint rule, (ii) trapezoid rule, (iii) Simpson rule (on the whole interval).

Find an upper bound for the error \( |Q - I| \leq \cdots \) using the error formulas in each case.

Here \( a = 0, b = 2 \) and \( f(x) = x^3 \), so we obtain

\[
Q_{\text{Midpt}} = 2 \cdot f(1) = 2, \quad Q_{\text{Trap}} = 2 \cdot \frac{f(0) + f(2)}{2} = 8, \quad Q_{\text{Simpson}} = 2 \cdot \frac{f(0) + 4 \cdot f(1) + f(2)}{6} = 4
\]

For the error estimates we have \( f''(x) = 6x \) and \( \max_{[0,2]} |f''(x)| = 12 \) yielding

\[
|Q_{\text{Midpt}} - I| \leq \frac{(b-a)^3}{24} \max_{[0,2]} |f''(x)| = \frac{2^3}{24} \cdot 12 = 4
\]
\[
|Q_{\text{Trap}} - I| \leq \frac{(b-a)^3}{12} \max_{[0,2]} |f''(x)| = \frac{2^3}{12} \cdot 12 = 8
\]
\[
|Q_{\text{Simpson}} - I| \leq \frac{(b-a)^5}{90 \cdot 32} \max_{[0,2]} |f^{(4)}(x)| = 0
\]

since we have \( f^{(4)}(x) = 0 \) in this case. Recall that the Simpson rule is actually exact if \( f(x) \) is a polynomial of degree \( \leq 3 \).

(b) Find the value of the composite trapezoid rule \( Q_{\text{Trap}}^2 \) with 2 subintervals of equal size.

Here \( N = 2 \) and \( h = (b-a)/N = 1 \) yielding

\[
Q_{\text{Trap}}^2 = h \cdot \frac{f(0) + f(1)}{2} = 1 \cdot 1 = 1
\]

(c) Find a value \( N \) such that we can guarantee \( \left| Q_{\text{Trap}}^2 - I \right| \leq 10^{-10} \) for the composite trapezoid rule with \( N \) intervals of equal size.

Using \( \max_{[0,2]} |f''(x)| = 12 \) we obtain

\[
\left| Q_{\text{Trap}}^2 - I \right| \leq \frac{1}{12} \frac{(b-a)^3}{N^2} \max_{x \in [a,b]} |f''(x)| = \frac{1}{12} \cdot \frac{2^3}{N^2} \cdot 12 = \frac{8}{N^2}
\]

We need to choose \( N \) such that

\[
\frac{8}{N^2} \leq 10^{-10} \iff \sqrt{\frac{8}{10^{-10}}} \leq N,
\]

i.e., for \( N \geq 10^5 \cdot \sqrt{8} \approx 282842.7 \) we will have an error \( \left| Q_{\text{Trap}}^2 - I \right| \leq 10^{-10} \).

2. Consider the initial value problem

\[
y'' + y' + y = t, \quad y(1) = 1, \quad y'(1) = 2
\]

(a) Let \( h = 1 \). Find an approximation for \( y(2) \) using one step of the (i) Euler method, (ii) improved Euler method.

(i) Let \( y_1 = y \) and \( y_2 = y' \). Then we get the system \( y' = f(t, y) \) with \( f(t, y) = \begin{bmatrix} y_2 \\ t-y_1-y_2 \end{bmatrix} \) and \( y(1) = y(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \). With \( s = f(1, y(0)) = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \), we obtain \( y^{(1)} = y^{(0)} + hs = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \), so the approximation for \( y(2) \) is 3.

(ii) We get \( s^{(1)} = f(1, y(1)) = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \) and \( s^{(2)} = f(2, y(0) + hs) = f(2, \begin{bmatrix} 3 \\ 0 \end{bmatrix}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \), so the approximation for \( y(2) \) is 2.

(b) Consider now the initial conditions \( y(1) = 1 \) and \( y'(1) = a \). We want to find \( b := y(2) \) for various values of \( a \).

Write a Matlab m-file \( \text{sol.m} \) starting with \textbf{function} \( \text{b=sol(a)} \) which has the input argument \( a \) and returns \( b := y(2) \).

\text{function} \( \text{b=sol(a)} \)
\text{f = @(t,y) [y(2);t-y(1)-y(2)];}
\text{[ts,ys]=ode45(f,[1,2],[1;a]);}
\text{b = ys(end,1); \quad % last row of ys for time T, first column for y1}
We try out the function \texttt{sol(a)} from (b) and obtain
\begin{verbatim}
>> [sol(-5), sol(0), sol(5)]
an = -1.5413 1.1262 3.7937
\end{verbatim}

We now want to find a solution \( y(t) \) of the ODE which satisfies \( y(1) = 1 \) and \( y(2) = 0 \). Write a Matlab program which plots this solution \( y(t) \). \textbf{Hint:} First find \( a \) such that \texttt{sol(a)} gives 0.

\begin{verbatim}
a = fzero (@sol, [-5, 0]); % find a in [-5,0] such that sol(a)=0
f = @(t,y) [y(2); t-y(1)-y(2)];
[ts, ys] = ode45(f, [1, 2], [1; a]); % solve IVP with y(1)=1, y'(1)=a
plot(ts, ys(:,1)) % plot first column of array ys
  % 1st column of ys gives values of y1(t)=y(t)
  % 2nd column of ys gives values of y2(t)=y'(t)
\end{verbatim}