1. We want to approximate a function value $y = f(x)$ with a Taylor polynomial $p_n(x) = f(x_0) + \cdots + f^{(n)}(x_0)(x-x_0)^n/n!$ of degree $n$. Find and evaluate the approximation $p_n(x)$. Find an upper bound $|f(x) - p_n(x)| \leq \cdots$ using the remainder term.

We can find an upper bound using

$$|f(x) - p_n(x)| = |R_{n+1}| = \frac{|x - x_0|^{n+1}}{(n+1)!} |f^{(n+1)}(t)| \leq \frac{|x - x_0|^{n+1}}{(n+1)!} \max_{t \in I} |f^{(n+1)}(t)|$$

where $I : = \min(x_0, x), \max(x_0, x)]$.

(a) $y = 32.0215$, $n = 1$
   $f(x) = x^{1/5}, f'(x) = \frac{1}{5}x^{-4/5}, f''(x) = -\frac{4}{25}x^{-9/5}$
   for $x_0 = 32$: $f(x_0) = 2, f'(x_0) = \frac{1}{5}, \frac{1}{16} = \frac{1}{80}$
   $p_1(x) = 2 + \frac{1}{80} \cdot (x - 32), p_1(32.02) = 2 + \frac{1}{80} \cdot 0.02 = 2.00025$
   $|y - p_1(x)| = |R_2| = \frac{1}{4} f''(t)(x - 32)^2 = \frac{1}{2} \cdot \frac{4}{25} \cdot 32^{-9/5} \cdot 0.02 \leq \frac{1}{2} \cdot \frac{4}{25} \cdot 32^{-9/5} \cdot 0.02 = 6.25 \cdot 10^{-8}$ (since $t^{-9/5}$ is decreasing on $[32, 32.02]$)

(b) $y = \cos 0.3$, $n = 5$
   $f(x) = f^{(4)}(x) = \cos x, f'(x) = f^{(5)}(x) = -\sin x, f''(x) = f^{(6)}(x) = -\cos x, f'''(x) = \sin x$
   for $x_0 = 0$: $f(0) = f^{(4)}(0) = 1, f'(0) = f^{(5)}(0) = 0, f''(0) = -1, f'''(0) = 0$
   $p_5(x) = 1 - \frac{1}{2} x^2 + \frac{1}{2} x^4, p_5(0.3) = 0.9553375, |y - p_5(x)| = |R_6| = \frac{1}{6!} |\cos t| \cdot 0.3^6 \leq \frac{1}{720} \cdot 1 \cdot 0.3^6 = 1.0125 \cdot 10^{-6}$

(c) $y = \ln(0.97)$, $n = 1$
   $f(x) = \ln x, f'(x) = x^{-1}, f''(x) = -x^{-2}$
   for $x_0 = 1$: $f(x_0) = 0, f'(x_0) = 1$
   $p_1(x) = x - 1, p_1(0.97) = -0.03$
   $|y - p_1(x)| = |R_2| = \frac{1}{2} |t^{-2}| \cdot (\cdot -0.3^2| \leq \frac{1}{2} \cdot 0.97^{-2} \cdot 0.3^2 \approx 4.78265 \cdot 10^{-2}$ (since $t^{-2}$ is decreasing on [.97, 1])

2. Let $x = 10^{-5}$. For each the following examples

   (i) $y = 3 - \sqrt{9 - x^2}$, (ii) $y = \log(1 - 2x)$, (iii) $y = \sin(1/x)$

   do the following:

   (a) Find the expression for the condition number $c_f(x)$ using pencil & paper. Evaluate the condition number using Matlab, print out the unavoidable error.

   (b) Compute the result using “naive evaluation” in Matlab (use format long g to show all digits). Find the relative error of the computed result by using extra precision arithmetic with vpa. Is the algorithm numerically stable?

   (i): condition number $c_f(x) = \frac{x^2}{x^2 - 9 + 3\sqrt{9 - x^2}} \bigg|_{x=10^{-5}} \approx 2.000035$, hence the unavoidable error is $\approx 3 \cdot 10^{-16}$.
   Naive evaluation gives $1.66666680456728 \cdot 10^{-11}$, the relative error is $8.3 \cdot 10^{-8}$. Since this is much larger than the unavoidable error, the algorithm is numerically unstable.

   (ii): $c_f(x) = \frac{-2x}{(1 - 2x) \cdot \log(1 - 2x)} \bigg|_{x=10^{-5}/3} \approx 1.000010000167$, unavoidable error $\approx 2\varepsilon_M$
   Naive evaluation gives $-2.00002000026867 \cdot 10^{-5}$, the relative error $\approx 10^{-12}$ is much larger than the unavoidable error, hence the algorithm is numerically unstable.

   (iii): $c_f(x) = \frac{\cos(x^{-1})}{x \cdot \sin(x^{-1})} \bigg|_{x=10^{-5}/3} \approx 2.8 \cdot 10^6$. The problem is very ill conditioned, and the unavoidable error is $|c_f(x)| \varepsilon_M + \varepsilon_M \approx 2.8 \cdot 10^{-10}$.
   Naive evaluation gives $0.0357487979865591$, the relative error is $4.07 \cdot 10^{-10}$. This is not much larger than the unavoidable error, hence the algorithm is numerically stable.
We are given \(b, c\) with \(b^2 - 4c \geq 0\). We want to find the solutions \(y_1, y_2\) (where \(y_1 \leq y_2\)) of the quadratic equation \(y^2 + by + c = 0\).

(a) Write a Matlab function \([y_1, y_2] = \text{qeq}(b, c)\) which computes the two solutions using “naive evaluation”.

For testing use
\[
\begin{align*}
z_1 &= 1e-7; \quad z_2 = 1/3; \quad b = -z_1 - z_2; \quad c = z_1 * z_2; \quad [y_1, y_2] = \text{qeq}(b, c);
\end{align*}
\]

and then compute the relative errors of the computed values \(y_1, y_2\) compared to \(z_1, z_2\).

Use (i) \(z_1 = 10^{-7}, \quad z_2 = \frac{1}{3}\); (ii) \(z_1 = -\frac{1}{3}, \quad z_2 = -10^{-7}\); (iii) \(z_1 = \frac{1}{3}, \quad z_2 = \frac{1}{3} + 10^{-7}\).

```matlab
function [y1, y2] = qeq(b, c)
s = sqrt(b^2 - 4*c);
y1 = (-b-s)/2;
y2 = (-b+s)/2;
end
```

(i): relative errors for \(y_1, y_2\): \(2.9 \times 10^{-11}, 0\)
(ii): relative errors for \(y_1, y_2\): \(0, 2.9 \times 10^{-11}\)
(iii): relative errors for \(y_1, y_2\): \(4.8 \times 10^{-10}, -4.8 \times 10^{-10}\)

Note that the errors are computed in machine arithmetic. Hence a value of 0 means that the actual relative error is of the order of \(10^{-16}\).

(b) For the exact value of \(b\) we obtain \(y_2 = f(b) = \frac{-b + \sqrt{b^2 - 4c}}{2}\). The computation uses a rounded value \(\hat{b}\) with \(\varepsilon_{\hat{b}} \leq \varepsilon_M\). Using \(\hat{b}\) instead of \(b\) gives \(\hat{y}_2 = f(\hat{b})\), with the relative error \(|\varepsilon_{\hat{y}_2}| \approx |c_f(b)| \varepsilon_{\hat{b}}|\). The condition number is
\[
c_f(b) = \frac{b \cdot f'(b)}{f(b)} = \frac{-b}{\sqrt{b^2 - 4c}}
\]

Note that for \(y_1 = \frac{-b - \sqrt{b^2 - 4c}}{2}\) we obtain the condition number \(\frac{b}{\sqrt{b^2 - 4c}}\), so the absolute value is the same. We now evaluate \(|c_f(b)| = \frac{|b|}{\sqrt{b^2 - 4c}}\) for (i), (ii), (iii):

(i): \(|c_f| \approx 1.0000000600000018\), unavoidable error\(\approx 2 \cdot 10^{-16}\). Computation of \(y_1\) was numerically unstable, computation of \(y_2\) was numerically stable.
(ii): \(|c_f| \approx 1.0000006000000018\), unavoidable error\(\approx 2 \cdot 10^{-16}\). Computation of \(y_1\) was numerically stable, computation of \(y_2\) was numerically unstable.
(iii): \(|c_f| \approx 6.687937001632672 \cdot 10^6\), unavoidable error\(\approx 10^{-10}\). Since the actual errors for \(y_1, y_2\) from (a) were not much larger than the unavoidable error, the computation was numerically stable for both \(y_1\) and \(y_2\).