Dynamic simulation of microbial growth (Linearized stability analysis based on eigenvalues.)
Instructor: Nam Sun Wang

Constitutive relations:

$\mu(s) := \dfrac{0.3 \cdot s}{50 + s}$ ... Monod specific growth rate

$Y(s) := 0.004 + 0.001 \cdot s$ ... yield coefficient

Dynamic Equations:

$dxdt\left(x, s, D, s_f\right) := (\mu(s) - D) \cdot x$

$dsdt\left(x, s, D, s_f\right) := D \cdot \left(s_f - s\right) - \dfrac{\mu(s)}{Y(s)} \cdot x$

Steady-states: (calculate by setting d/dt=0 with initial guesses: $x := 1$ $s := 0$

Note that with an initial guess of x=0, computer gives the washout steady-state, which leads to one positive and one negative eigenvalues. Thus, the washout steady-state is unstable. With an initial guess of x=1, computer gives the non-washout steady-state. If the non-washout steady-state is also unstable, we have a limit cycle.

Given $dxdt\left(x, s, D, s_f\right) = 0$

$dsdt\left(x, s, D, s_f\right) = 0$

$ss\left(D, s_f\right) := Find(x, s)$

The 0th answer is the biomass steady-state: $x_{ss}\left(D, s_f\right) := ss\left(D, s_f\right)_0$

The 1st answer is the substrate steady-state: $s_{ss}\left(D, s_f\right) := ss\left(D, s_f\right)_1$

An example: $ss(0.1, 200) = \begin{pmatrix} 5.075 \\ 25 \end{pmatrix}$

Form the Jacobian matrix such that dX/dt=AX, where X is the deviation variable. (Evaluate d/dx and d/ds with |Symbolic|Differentiate on Variable| on dynamic equation.)

$$A\left(x, s, D, s_f\right) := \begin{bmatrix} \mu(s) - D & \dfrac{d}{ds}\mu(s) \cdot x \\ \dfrac{-\mu(s)}{Y(s)} & -D - \dfrac{\dfrac{d}{ds}\mu(s)}{Y(s)} \cdot x + \dfrac{\mu(s)}{Y(s)^2} \cdot x \cdot \dfrac{d}{ds}Y(s) \end{bmatrix}$$

Find the eigenvalue of matrix A, evaluated at steady-state points

$\lambda\left(D, s_f\right) := eigenvals\left(A\left(x_{ss}\left(D, s_f\right), s_{ss}\left(D, s_f\right), D, s_f\right)\right)$

An example:

$\lambda(0.1, 200) = \begin{pmatrix} 0.01839 + 0.21524i \\ 0.01839 - 0.21524i \end{pmatrix}$

Positive real part shows that the non-washout steady-state is unstable, and the imaginary part shows spiral.

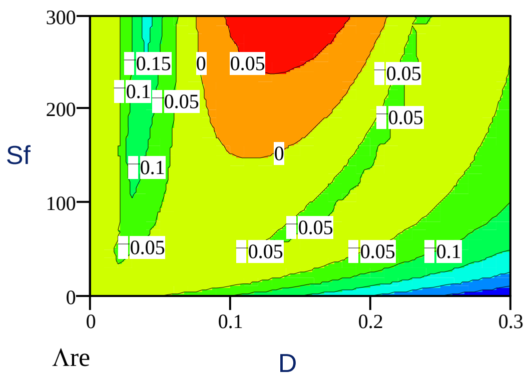
Range operating conditions D and sf to see how the eigenvalue changes at each point

$i := 0..30$ $D_i := 0.01 \cdot i$ ... Dilution rate

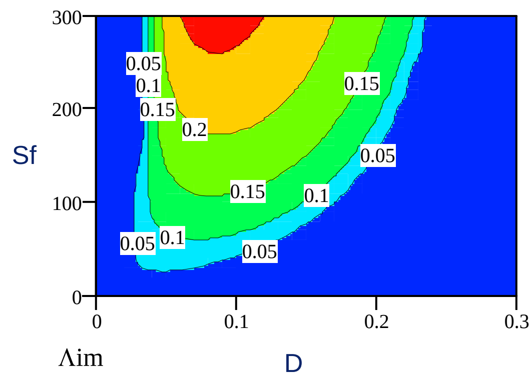
$j := 0..30$ $s_{f_j} := 10 \cdot j$ ... Feed substrate concentration

$\Lambda re_{i,j} := \mathrm{Re}\left(\lambda\left(D_i, s_{f_j}\right)_0\right)$ $\Lambda im_{i,j} := \mathrm{Im}\left(\lambda\left(D_i, s_{f_j}\right)_0\right)$

Real part: stable | unstable



Imaginary part: exponential | oscillation



Combine the above two steady-state stability plot into one plot with 4 stability regions:

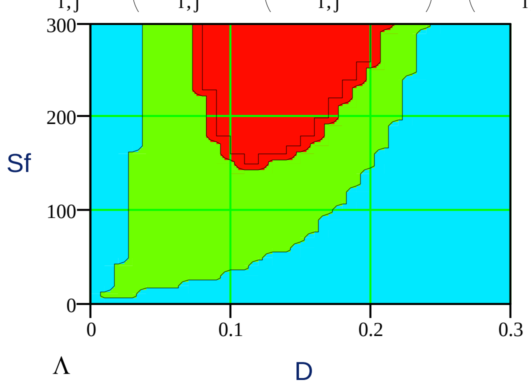
0 ... Real Part < 0 & Imaginary part = 0 (stable exponential approach)

1 ... Real Part < 0 & Imaginary part <> 0 (stable oscillatory approach)

2 ... Real Part > 0 & Imaginary part <> 0 (unstable oscillatory approach)

3 ... Real Part > 0 & Imaginary part = 0 (unstable exponential approach)

$\Lambda_{i,j} := \mathrm{if}\left(\Lambda re_{i,j} \le 0, \mathrm{if}\left(\Lambda im_{i,j} \ne 0, 1, 0\right), \mathrm{if}\left(\Lambda im_{i,j} \ne 0, 2, 3\right)\right)$



Dynamic simulation with initial conditions $x_0 := 0.5$ $s_0 := 30$

and operating conditions $D := 0.1$ $s_f := 200$

$$ydot(t,y) := \begin{pmatrix} dxdt(y_0, y_1, D, s_f) \\ dsdt(y_0, y_1, D, s_f) \end{pmatrix} \qquad y_{initial} := \begin{pmatrix} x_0 \\ s_0 \end{pmatrix}$$

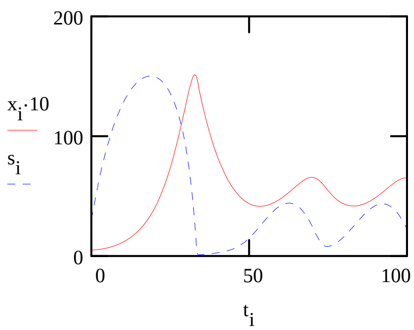
... biomass

... substrate

Solve both sets of ODEs (integrate from $t_0 := 0$ to $t_f := 100$ in $nstep := 1000$

$yout := rkfixed(y_{initial}, t_0, t_f, nstep, ydot)$ $t := yout^{\langle 0 \rangle}$ $x := yout^{\langle 1 \rangle}$ $s := yout^{\langle 2 \rangle}$

Plots of state variables $i := 0 .. last(t)$



Phase diagram