Find the steady-state values for a continuous bioreactor with **cell recycle**.
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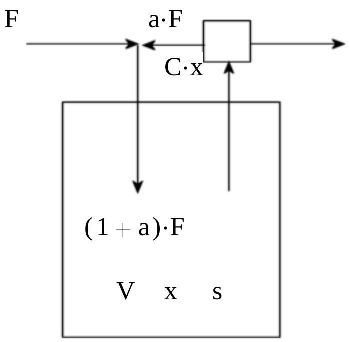


F ... flow rate

a ... recycle ratio (Note that textbook uses the symbol α, which denotes growth related product formation here.)

C ... biomass concentration factor

V ... bioreactor volume

x, s ... biomass & substrate concentration

Given Steady-state conditions:

dx/dt=0 $0 = (\mu - (1 + a - a \cdot C) \cdot D) \cdot x$ ... Biomass

ds/dt=0 $0 = D \cdot (s_f - s) - \frac{1}{Yx} \cdot \mu \cdot x$ ... Substrate

where (constitutive relationship) $\mu = \frac{\mu_m \cdot s}{K + s}$

Turn on |Math|SmartMath| to let MathCAD evaluate the solution to the above problem. There is no need to give an initial guess of (x, s). Since none of the model parameters has been specified, "Find" will find the analytical solution in terms of these unspecified variables. There are three equations; thus, we can choose to solve for any of the three variables in terms of others.

$$\text{Find}(x, s, \mu) \rightarrow \begin{bmatrix} 0 & -Yx \cdot \frac{(-D \cdot s_f + D \cdot s_f \cdot a \cdot C + \mu_m \cdot s_f + D \cdot K \cdot a \cdot C - D \cdot s_f \cdot a - K \cdot D - D \cdot K \cdot a)}{(2 \cdot D \cdot a - 2 \cdot D \cdot a \cdot C + D \cdot a^2 - 2 \cdot D \cdot a^2 \cdot C + D \cdot a^2 \cdot C^2 - \mu_m - \mu_m \cdot a + \mu_m \cdot a \cdot C + D)} \\ s_f & -(-1 - a + a \cdot C) \cdot K \cdot \frac{D}{(D \cdot a \cdot C - D \cdot a + \mu_m - D)} \\ \mu_m \cdot \frac{s_f}{(K + s_f)} & D + D \cdot a - D \cdot a \cdot C \end{bmatrix}$$

Note the first vector is the washout steady-state, and the second vector is the nontrivial solution. These expressions can be further simplified by collecting terms, etc.

The following includes product.

Given

dx/dt=0 $0=(\mu-(1+a-a\cdot C)\cdot D)\cdot x$ ... Biomass

ds/dt=0 $0=D\cdot\left(s_f-s\right)-\frac{1}{Yx}\cdot\mu\cdot x-\frac{1}{Yp}\cdot(\alpha\cdot\mu\cdot x+\beta\cdot x)$ ... Substrate

dp/dt=0 $0=\alpha\cdot\mu\cdot x+\beta\cdot x-D\cdot p$ ... Product

where $\mu=\frac{\mu_m\cdot s}{K+s}$

$$\mathrm{Find}(x,s,p,\mu)\rightarrow\begin{bmatrix} 0 & -D\cdot Yx\cdot Yp\cdot\frac{\quad}{\left(-2\cdot Yx\cdot\alpha\cdot D^2\cdot a^2\cdot C-2\cdot Yx\cdot\alpha\cdot D^2\cdot a\cdot C+Yx\cdot\alpha\cdot D^2\cdot a^2\cdot C\right.} \\ s_f & \\ 0 & Yx\cdot Yp\cdot\left(-D\cdot s_f+D\cdot s_f\cdot a\cdot C+\mu_m\cdot s_f+D\cdot K\cdot a\cdot C-D\cdot s_f\cdot a-K\cdot D-D\cdot K\cdot a\right)\cdot\frac{\quad}{\left(-2\cdot Yx\cdot\alpha\cdot D^2\cdot a^2\cdot C-\right.} \\ \mu_m\cdot\frac{s_f}{\left(K+s_f\right)} & \end{bmatrix}$$

Numerical Solution

Express the biomass/substrate/product concentrations at steady-state as functions of D and s_f (by copying the formula from above). Washout at $\mu=(1+a-a\cdot C)\cdot D=\mu_m\cdot s_f/(K+s_f)$

Without the product term.

$Yx := 0.5 \qquad \mu_m := 0.5 \qquad K := 1 \qquad s_f := 1$

$$s(D, s_f, a, C) := \mathrm{if}\left[D \geq \frac{1}{1+a-a\cdot C}\cdot\frac{\mu_m\cdot s_f}{K+s_f}, s_f, \frac{(1+a-a\cdot C)\cdot D\cdot K}{\mu_m-(1+a-a\cdot C)\cdot D}\right]$$

$$x(D, s_f, a, C) := \mathrm{if}\left[D \geq \frac{1}{1+a-a\cdot C}\cdot\frac{\mu_m\cdot s_f}{K+s_f}, 0, \frac{Yx\cdot\left(s_f-s(D, s_f, a, C)\right)}{1+a-a\cdot C}\right]$$

Comparison of steady-state variables /w and /wo cell recycle $\qquad D := 0.01, 0.02 .. \mu_m$

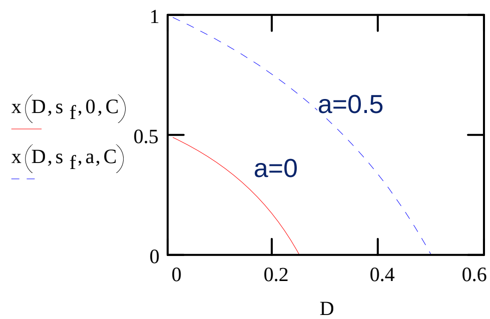
fraction of recycle flow

$a := 0.5$

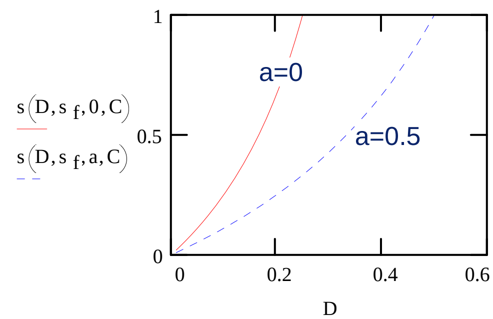
cell concentration factor

$C := 2$

Biomass Concentration

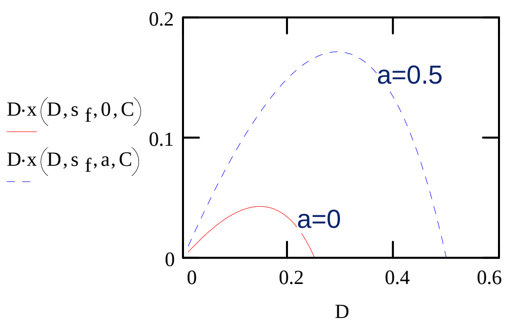


Substrate Concentration



Comparison of steady-state cell/product productivities /w and /wo cell recycle

Cell productivity



With the product term.

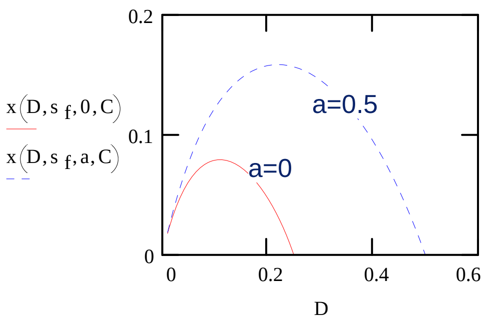
$Yp := 0.2 \qquad \alpha := 0.5 \qquad \beta := 0.1$

$$s(D, s_f, a, C) := \text{if}\left[D \geq \frac{1}{1+a-a\cdot C}\cdot\frac{\mu_m \cdot s_f}{K+s_f}, s_f, \frac{(1+a-a\cdot C)\cdot D\cdot K}{\mu_m-(1+a-a\cdot C)\cdot D}\right]$$

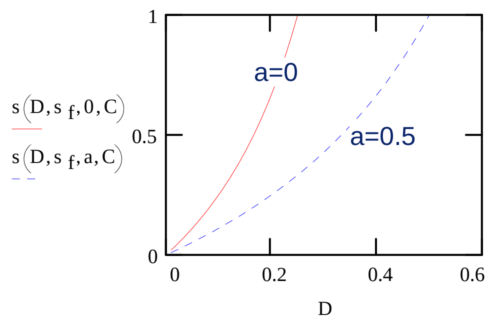
$$x(D, s_f, a, C) := \text{if}\left[D \geq \frac{1}{1+a-a\cdot C}\cdot\frac{\mu_m \cdot s_f}{K+s_f}, 0, \frac{D\cdot Yx\cdot Yp\cdot\left(s_f - s(D, s_f, a, C)\right)}{D\cdot(1+a-a\cdot C)\cdot(Yp+\alpha\cdot Yx)+\beta\cdot Yx}\right]$$

$$p(D, s_f, a, C) := \text{if}\left[D \geq \frac{1}{1+a-a\cdot C}\cdot\frac{\mu_m \cdot s_f}{K+s_f}, 0, \left[\alpha\cdot(1+a-a\cdot C)+\frac{\beta}{D}\right]\cdot x(D, s_f, a, C)\right]$$

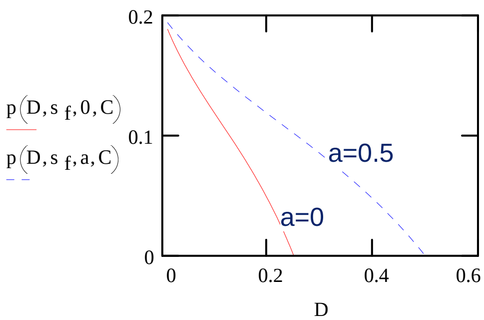
Biomass Concentration



Substrate Concentration

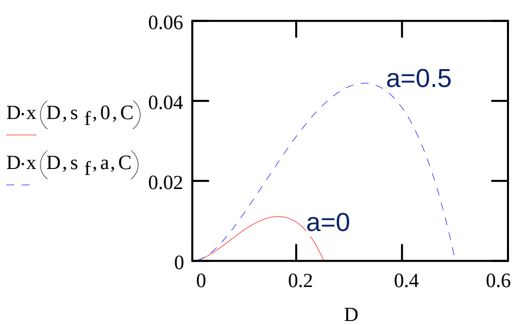


Product Concentration

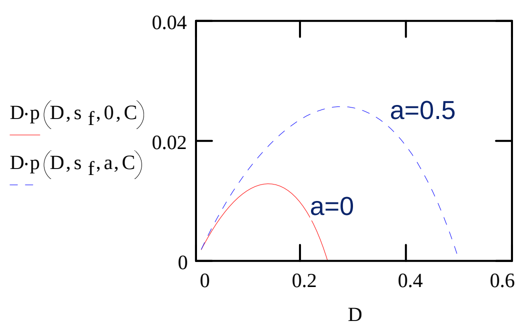


Comparison of steady-state cell/product productivities /w and /wo cell recycle

Cell productivity



Product Productivity



Summary. Depending on whether part of the substrate is diverted to product, the biomass versus dilution rate plot appear somewhat differently toward D=0.
It is clear from the graph that both cell and product concentrations are higher with cell recycle. Intuitively, cell recycle always increase the level of cells in a bioreactor. In turn, more cells yield more product. The same can be said of cell and product productivities.

$$\left(-D\cdot s_f + D\cdot s_f\cdot a\cdot C + \mu_m\cdot s_f + D\cdot K\cdot a\cdot C - D\cdot s_f\cdot a - K\cdot D -\right.$$

$$^2 - Yx\cdot\beta\cdot D\cdot a\cdot C + \mu_m\cdot Yp\cdot D\cdot a\cdot C - \mu_m\cdot Yp\cdot D\cdot a - \mu_m\cdot Yx\cdot\alpha\cdot D + \mu_m\cdot Yx\cdot\alpha\cdot D\cdot a\cdot C - \mu_m\cdot Yx\cdot\alpha\cdot D\cdot a + Yx\cdot\beta\cdot D\cdot a + Yx\cdot\alpha\cdot D^2\cdot a^2 + 2\cdot Yx\cdot$$

$$-(-1 - a + a\cdot C)\cdot K\cdot\frac{D}{(D\cdot a\cdot C - D\cdot a + \mu_m - D)}$$

$$(-D\cdot\alpha - \alpha$$

$$2\cdot Yx\cdot\alpha\cdot D^2\cdot a\cdot C + Yx\cdot\alpha\cdot D^2\cdot a^2\cdot C^2 - Yx\cdot\beta\cdot D\cdot a\cdot C + \mu_m\cdot Yp\cdot D\cdot a\cdot C - \mu_m\cdot Yp\cdot D\cdot a - \mu_m\cdot Yx\cdot\alpha\cdot D + \mu_m\cdot Yx\cdot\alpha\cdot D\cdot a\cdot C - \mu_m\cdot Yx\cdot\alpha\cdot D\cdot a + Y$$

$$D + D\cdot a - D\cdot a\cdot C$$

$$+ Yx\cdot\alpha\cdot D^2 + Yx\cdot\beta\cdot D + Yp\cdot D^2\cdot a^2\cdot C^2 - 2\cdot Yp\cdot D^2\cdot a^2\cdot C + Yp\cdot D^2 + Yp\cdot D^2\cdot a^2 - 2\cdot Yp\cdot D^2\cdot a\cdot C + 2\cdot Yp\cdot D^2\cdot a - \mu_m\cdot Yx\cdot\beta\Big)$$

$$)$$

$$\cdot a^2 + 2\cdot Yx\cdot\alpha\cdot D^2\cdot a - \mu_m\cdot Yp\cdot D + Yx\cdot\alpha\cdot D^2 + Yx\cdot\beta\cdot D + Yp\cdot D^2\cdot a^2\cdot C^2 - 2\cdot Yp\cdot D^2\cdot a^2\cdot C + Yp\cdot D^2 + Yp\cdot D^2\cdot a^2 - 2\cdot Yp\cdot D^2\cdot a\cdot C + 2\cdot Yp\cdot D^2\cdot a$$