Approximate a function within the given interval with orthogonal polynomials & Bessel functions.
With formula from Tuma, Jan J., *Engineering Mathematics Handbook*, McGraw-Hill, 1979.
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Given Function defined over x=[x_{lo},x_{hi}] $x_{lo} := 0$ $x_{hi} := 5$

$f(x) := x \cdot e^{-x}$ ← Change this function to see how other functions (sin(x), cos(x), or whatever other function you desire) can be approximated. The rest of the worksheet below can be left untouched.

Number of Terms Used: $N := 4$

0. Taylor's Series Expansion (which is the reference for comparison) around $x := 0$ for $i := 0 .. N$

$$a_i := \frac{d^i}{dx^i} f(x) \qquad a^T = (\,0 \quad 1 \quad -2 \quad 3 \quad -4\,) \qquad f_{Taylor}(x) := \sum_{i=0}^{N} \frac{a_i}{i!} \cdot x^i$$

I. Legendre Series -- Legendre polynomials are orthogonal over z=[-1,1]

$$P_0(z) := 1 \quad P_1(z) := z \quad P_2(z) := \frac{1}{2} \cdot \left(3 \cdot z^2 - 1\right) \quad P_3(z) := \frac{1}{2} \cdot \left(5 \cdot z^3 - 3 \cdot z\right) \quad P_4(z) := \frac{1}{8} \cdot \left(35 \cdot z^4 - 30 \cdot z^2 + 3\right)$$

Legendre polynomials in a vector form: $P(z) := \left(P_0(z) \;\; P_1(z) \;\; P_2(z) \;\; P_3(z) \;\; P_4(z)\right)^T$

Compress/expand x=[x_{lo},x_{hi}] into z=[-1,1] $z_{lo} := -1$ $z_{hi} := 1$

$$\alpha 0 := \frac{x_{lo} \cdot z_{hi} - z_{lo} \cdot x_{hi}}{x_{lo} - x_{hi}} \qquad \beta 0 := \frac{z_{hi} - z_{lo}}{x_{hi} - x_{lo}} \qquad \alpha 0 = -1 \qquad \beta 0 = 0.4 \qquad zz(x) := \alpha 0 + \beta 0 \cdot x$$

$$\alpha 1 := \frac{x_{lo} \cdot z_{hi} - z_{lo} \cdot x_{hi}}{z_{hi} - z_{lo}} \qquad \beta 1 := \frac{x_{hi} - x_{lo}}{z_{hi} - z_{lo}} \qquad \alpha 1 = 2.5 \qquad \beta 1 = 2.5 \qquad xx(z) := \alpha 1 + \beta 1 \cdot z$$

The same function in z-variable space: $F(z) := f(xx(z))$

The best approximation of f in the space spanned by the Legendre polynomials (or any set of basis functions) is the **projection** of the given function f into the Legendre polynomial space, which is the inner product between these two functions.

$$b_i := \frac{2 \cdot i + 1}{2} \cdot \int_{-1}^{1} F(z) \cdot P(z)_i \, dz \qquad f_{Legendre}(z) := \sum_{i=0}^{N} b_i \cdot P(z)_i$$

II. Chebyshev Series -- Chebyshev polynomials are orthogonal over z=[1,1]

$$T_0(z) := 1 \quad T_1(z) := z \quad T_2(z) := 2 \cdot z^2 - 1 \quad T_3(z) := 4 \cdot z^3 - 3 \cdot z \quad T_4(z) := 8 \cdot z^4 - 8 \cdot z^2 + 1$$

Chebyshev polynomials in a vector form: $T(z) := \left(T_0(z) \;\; T_1(z) \;\; T_2(z) \;\; T_3(z) \;\; T_4(z)\right)^T$

Function approximation by the Chebychev polynomials:

$$c_i := \int_{-1}^{1} F(z) \cdot T(z)_i \, dz$$

Add a small value of ε to avoid dividing by 0 at z=±1. $\varepsilon := 10^{-6}$

$$f_{Chebychev}(z) := \frac{c_0}{\pi \cdot \sqrt{1 + \varepsilon - z^2}} + \frac{2}{\pi \cdot \sqrt{1 + \varepsilon - z^2}} \cdot \sum_{i=1}^{N} c_i \cdot T(z)_i$$

III. Bessel Series -- Bessel functions are orthogonal over z=[0,1]

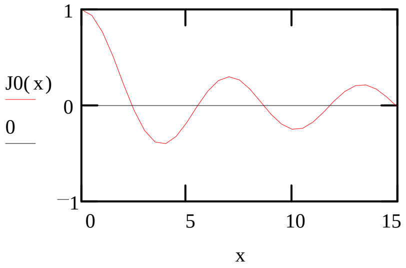
Compress/expand x=[x_{lo},x_{hi}] into z=[0,1] $z_{lo} := 0$ $z_{hi} := 1$

$$\alpha0 := \frac{x_{lo} \cdot z_{hi} - z_{lo} \cdot x_{hi}}{x_{lo} - x_{hi}} \quad \beta0 := \frac{z_{hi} - z_{lo}}{x_{hi} - x_{lo}} \quad \alpha0 = 0 \quad \beta0 = 0.2 \quad zBel(x) := \alpha0 + \beta0 \cdot x$$

$$\alpha1 := \frac{x_{lo} \cdot z_{hi} - z_{lo} \cdot x_{hi}}{z_{hi} - z_{lo}} \quad \beta1 := \frac{x_{hi} - x_{lo}}{z_{hi} - z_{lo}} \quad \alpha1 = 0 \quad \beta1 = 5 \quad xBel(z) := \alpha1 + \beta1 \cdot z$$

The same function in z-variable space: $F(z) := f(xBel(z))$

Function approximation by Bessel Function of the First Kind of order 0: $x := 0, 0.5 .. 15$



The roots of J0(x):

$rootJ0(x) := root(J0(x), x)$

$i := 0 .. 8$ Temporarily increase the number of terms.

$\lambda_0 := rootJ0(2.5)$

$\lambda_{i+1} := rootJ0(\lambda_i + \pi)$

$\lambda^T =$

	0	1	2	3	4	5	6	7
0	2.405	5.52	8.654	11.792	14.931	18.071	21.212	24.352

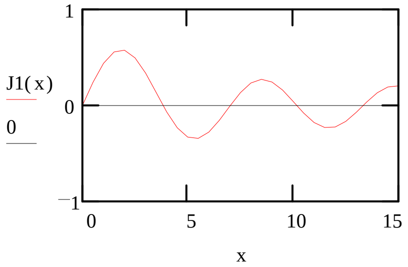
$$d0_i := \frac{2}{J1(\lambda_i)^2} \cdot \int_0^1 z \cdot F(z) \cdot J0(\lambda_i \cdot z)\, dz \qquad f_{Bessel0}(z) := \sum_{i=0}^{N} d0_i \cdot J0(\lambda_i \cdot z)$$

$$f_{Bessel08}(z) := \sum_{i=0}^{8} d0_i \cdot J0(\lambda_i \cdot z)$$

Restore the number of terms.

$i := 0 .. N$

Function approximation by Bessel Function of the First Kind of order 1:



The roots of J1(x):

$rootJ1(x) := root(J1(x), x)$

$\lambda_0 := rootJ1(\pi)$

$\lambda_{i+1} := rootJ1(\lambda_i + \pi)$

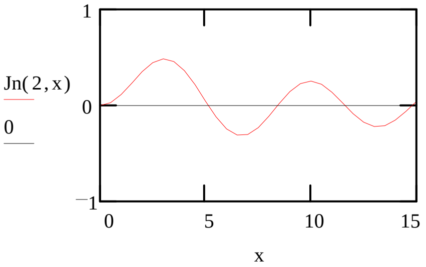
$\lambda^T =$

	0	1	2	3	4	5	6
0	3.833	7.016	10.173	13.324	16.471	19.616	21.212

$$d1_i := \frac{2}{Jn(2, \lambda_i)^2} \cdot \int_0^1 z \cdot F(z) \cdot J1(\lambda_i \cdot z)\, dz \qquad f_{Bessel1}(z) := \sum_{i=0}^{N} d1_i \cdot J1(\lambda_i \cdot z)$$

Function approximation by Bessel Function of the First Kind of order 2: $n := 2$

The roots of J2(x):



$$\mathrm{rootJn}(n,x) := \mathrm{root}(\mathrm{Jn}(n,x),x)$$

$$\lambda_0 := \mathrm{rootJn}(n,5)$$

$$\lambda_{i+1} := \mathrm{rootJn}\left(n,\lambda_i+\pi\right)$$

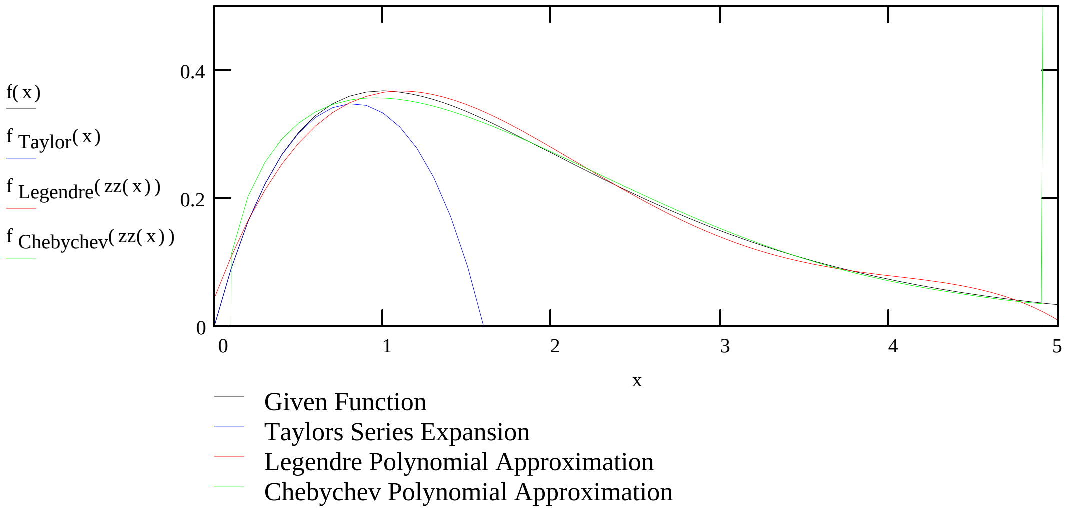
$\lambda^T =$

	0	1	2	3	4	5	6	7
0	5.135	8.417	11.62	14.796	17.96	21.117	21.212	24.352

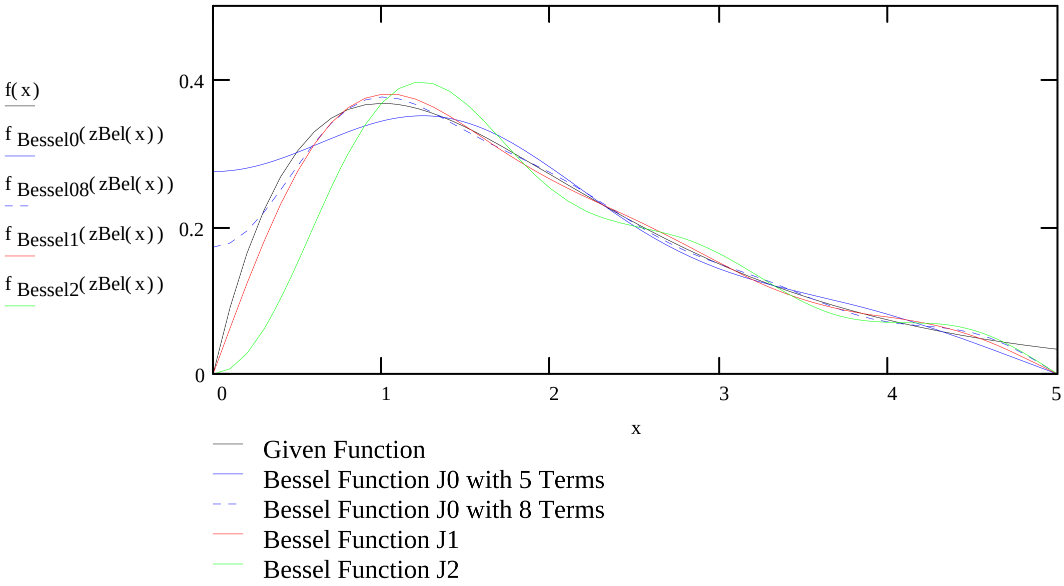
$$d2_i := \frac{2}{\mathrm{Jn}\left(3,\lambda_i\right)^2}\cdot\int_0^1 z\cdot F(z)\cdot\mathrm{Jn}\left(2,\lambda_i\cdot z\right)\,dz \qquad f_{\mathrm{Bessel2}}(z) := \sum_{i=0}^{N} d2_i\cdot\mathrm{Jn}\left(2,\lambda_i\cdot z\right)$$

Comparison of Taylor's series expansion and the various orthogonal polynomial approximations over $x=[x_{lo},x_{hi}]$

$$x := x_{lo}, x_{lo} + 0.1 .. x_{hi}$$



Comparison of approximation with various Bessel functions over $x=[x_{lo},x_{hi}]$



We can see from the above plots that the Taylor's series expansion is valid only in the close vicinity of x=0 and quickly deviates from the original function that it is trying to approximate. On the other hand, approximation with orthogonal Legendre polynomials of the same degree as the Taylor's series is quite good over the entire range. Chebychev polynomials also do just as well. The various Bessel functions do not perform as well. Approximation with J0, in particular, is not very good. There is less error when we increase the number of terms to 8.