

Lagrange Polynomial Interpolation. -- Normal Distribution Function $\exp(-x^2)$, which is given first.
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Example: **Normal Distribution Function** defined over $x=[a,b]=[-2,2]$ with 9 points.

$$f(x) := \exp(-x^2)$$

$N := 8$ $j := 0..N$... number of interpolation points

$a := -2$ $b := 2$... two end points

Lagrange Polynomial Interpolation

Specify non-equally spaced points $j=0..N$ over an interval of $[a,b]$ based on **Chebyshev's formula**

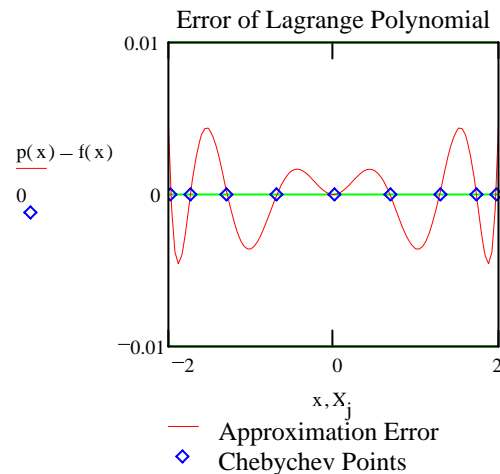
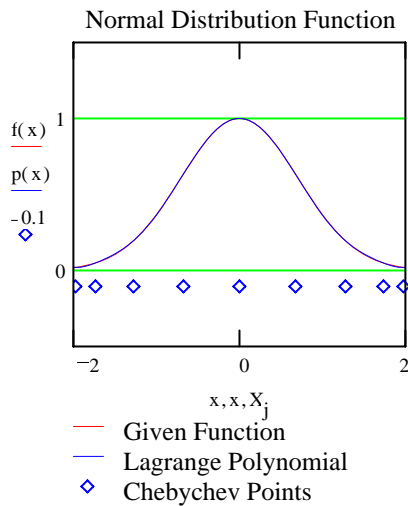
$$X_j := \frac{a+b}{2} + \frac{a-b}{2} \cdot \cos\left[\frac{2 \cdot j + 1}{2 \cdot (N+1)} \cdot \pi\right]$$

Lagrange's interpolating polynomial. $f(X_j)$ are values of f evaluated at the Chebyshev points.

$$Y_j := \frac{f(X_j)}{\prod_{k=0}^N \text{if}(k \neq j, X_j - X_k, 1)}$$

$$p(x) := \left[\prod_{j=0}^N (x - X_j) \right] \cdot \left(\sum_{j=0}^N \frac{Y_j}{x - X_j} \right)$$

Compare $f(x)$ against Lagrange's polynomial interpolation formula p $x := a, a + 0.01 \cdot (b - a) .. b$



Sum of squared error (sse):

$$\text{sse} := \int_{-1}^1 (p(x) - f(x))^2 dx$$

$$\text{sse} = 5.083 \cdot 10^{-6}$$

Note: The approximation is good, with almost no discernible difference between the original given function $f(x)$ and the interpolating function $p(x)$. Specifically, there is no error at the Chebyshev points.

Interpolation Based on Equally Spaced Interpolation Points

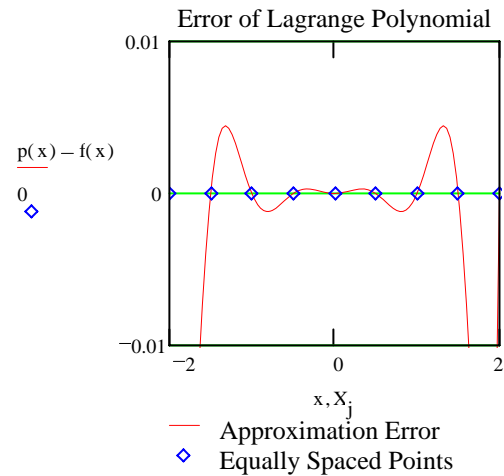
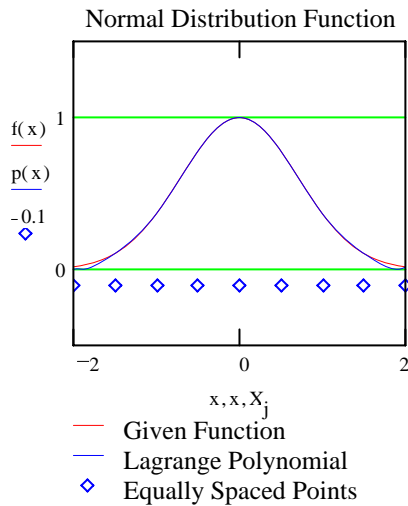
$$X_j := a + (b - a) \cdot \left(\frac{j}{N}\right)$$

Lagrange's interpolating polynomial. $f(X_j)$ are values of f evaluated at equally spaced points.

$$Y_j := \frac{f(X_j)}{\prod_{k=0}^{N-1} \text{if}(k \neq j, X_j - X_k, 1)}$$

$$p(x) := \left[\prod_{j=0}^{N-1} (x - X_j) \right] \cdot \left(\sum_{j=0}^{N-1} \frac{Y_j}{x - X_j} \right)$$

Compare $f(x)$ against Lagrange's polynomial interpolation formula p



Sum of squared error (sse):

$$\text{sse} := \int_{-1}^1 (p(x) - f(x))^2 dx$$

$$\text{sse} = 3.5 \cdot 10^{-4}$$

Note: There is visible error at both ends of the interpolation function. As before, there is no error at the equally spaced interpolation points. The sse value is about 100 times larger in this case, compared to interpolation with Chebychev points.

Least Squares Approximation of $f(x)$ with nine basis functions $1, x, x^2, \dots, x^8$. (Actually, the odd powered terms are zero because $f(x)$ is symmetric.)

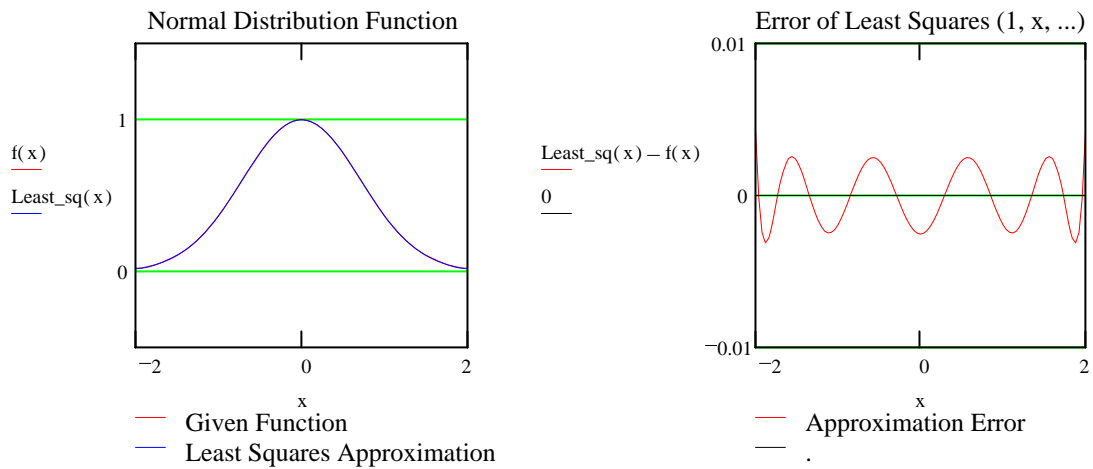
Evaluate f at 101 points: $i := 0..100$ $xx_i := a + \frac{b-a}{100} \cdot i$

$$X_{i,j} := (xx_i)^j \quad y_i := f(xx_i)$$

Least squares formula:

$$c := (X^T \cdot X)^{-1} \cdot X^T \cdot y \quad \text{Least_sq}(x) := \sum_{j=0}^N c_j \cdot x^j$$

Compare $f(x)$ against Least squares approximation.



Sum of squared error (sse):

$$\text{sse} := \int_{-1}^1 (\text{Least_sq}(x) - f(x))^2 dx$$

$$\text{sse} = 5.787 \cdot 10^{-6}$$

Note: The error, which is not visible in the $f(x)$ plot, is evenly distributed throughout the given interval. The sse value is comparable to the Lagrange interpolation polynomial with Chebychev points. There are 10 points at which there is no error.

Least Squares Approximation of $f(x)$ with five basis functions $1, \cos(x), \cos(2x), \dots, \cos(5x)$. (We use five terms to make a fair comparison because the odd powered terms are missing previously. These correspond to the sine terms that are missing here.)

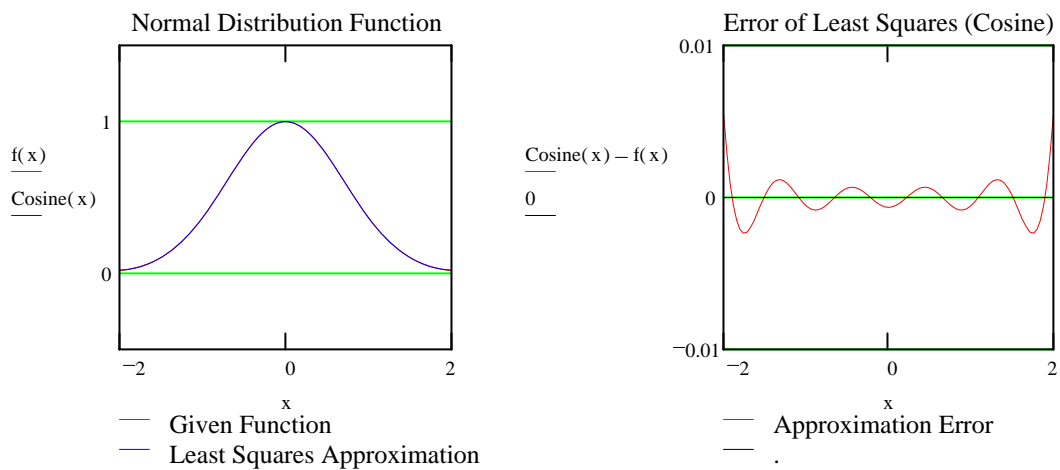
Evaluate f at 101 points: $j := 0..4$

$$XX_{i,j} := \cos\left(\frac{2 \cdot \pi}{b-a} \cdot j \cdot xx_i\right) \quad yy_i := f(xx_i)$$

Least squares formula:

$$d := (XX^T \cdot XX)^{-1} \cdot XX^T \cdot yy \quad \text{Cosine}(x) := \sum_{j=0}^4 d_j \cdot \cos\left(\frac{2 \cdot \pi}{b-a} \cdot j \cdot x\right)$$

Compare $f(x)$ against Least squares approximation



Sum of squared error (sse):

$$\text{sse} := \int_{-1}^1 (\text{Cosine}(x) - f(x))^2 dx$$

$$\text{sse} = 5.642 \cdot 10^{-7}$$

Note: The error, which is not visible in the $f(x)$ plot, is evenly distributed throughout the given interval. The sse value is comparable to the Lagrange interpolation polynomial with Chebychev points.

Taylor's Series Expansion around $x=0$ with the same number of terms as above

$$c_j := \frac{d^j}{d x^j} f(x) \quad \leftarrow \text{resulted in "Invalid order" error -- Mathcad could go up to only } j=5.$$

Otherwise, this is the easier way of evaluating the derivatives.

Hence, we mark "x" and choose from menu |Symbolic|Differentiate on Variable| successively.

$$f_0(x) := \exp(-x^2)$$

$$f_1(x) := -2 \cdot x \cdot \exp(-x^2)$$

$$f_2(x) := -2 \cdot \exp(-x^2) + 4 \cdot x^2 \cdot \exp(-x^2)$$

$$f_3(x) := 12 \cdot x \cdot \exp(-x^2) - 8 \cdot x^3 \cdot \exp(-x^2)$$

$$f_4(x) := 12 \cdot \exp(-x^2) - 48 \cdot x^2 \cdot \exp(-x^2) + 16 \cdot x^4 \cdot \exp(-x^2)$$

$$f_5(x) := -120 \cdot x \cdot \exp(-x^2) + 160 \cdot x^3 \cdot \exp(-x^2) - 32 \cdot x^5 \cdot \exp(-x^2)$$

$$f_6(x) := -120 \cdot \exp(-x^2) + 720 \cdot x^2 \cdot \exp(-x^2) - 480 \cdot x^4 \cdot \exp(-x^2) + 64 \cdot x^6 \cdot \exp(-x^2)$$

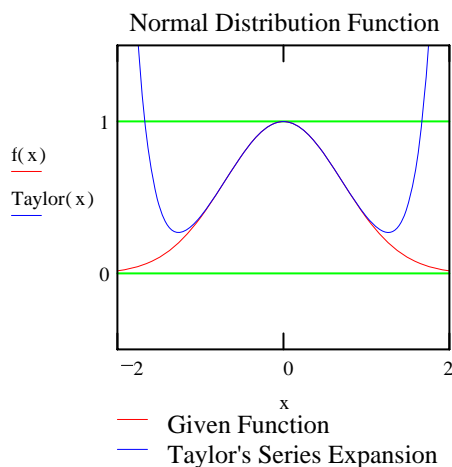
$$f_7(x) := 1680 \cdot x \cdot \exp(-x^2) - 3360 \cdot x^3 \cdot \exp(-x^2) + 1344 \cdot x^5 \cdot \exp(-x^2) - 128 \cdot x^7 \cdot \exp(-x^2)$$

$$f_8(x) := 1680 \cdot \exp(-x^2) - 13440 \cdot x^2 \cdot \exp(-x^2) + 13440 \cdot x^4 \cdot \exp(-x^2) - 3584 \cdot x^6 \cdot \exp(-x^2) + 256 \cdot x^8 \cdot \exp(-x^2)$$

$$c := (f_0(0) \ f_1(0) \ f_2(0) \ f_3(0) \ f_4(0) \ f_5(0) \ f_6(0) \ f_7(0) \ f_8(0))^\top$$

$$c^\top = (1 \ 0 \ -2 \ 0 \ 12 \ 0 \ -120 \ 0 \ 1.68 \cdot 10^3)$$

$$\text{Taylor}(x) := \sum_{j=0}^N \frac{c_j}{j!} \cdot x^j$$



Taylor's series expansion is valid only near the point of expansion (i.e., around $x=0$). Away from this point, the approximation fails quickly and very badly.