Eigenvalue-Eigenvector
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Any vector x in real Euclidean space of dimension n can be uniquely expressed as a linear combination of n linearly independent vectors (i.e., basis) g_j, j=1,2,..., n.

$$x = \alpha_1 \cdot g_1 + \alpha_2 \cdot g_2 + \dots + \alpha_n \cdot g_n = \sum_{j=1}^{n} \alpha_j \cdot g_j$$

Thus, given a linear transformation $\mathcal{A}$, we can find the action of $\mathcal{A}$ on any of the vectors x that reside in the same real Euclidean space if we know the action of $\mathcal{A}$ on each of the n linearly independent basis vectors g_i.

$$\mathcal{A} \cdot x = \alpha_1 \cdot (\mathcal{A} \cdot g_1) + \alpha_2 \cdot (\mathcal{A} \cdot g_2) + \dots + \alpha_n \cdot (\mathcal{A} \cdot g_n)$$

An **invariant subspace** for linear transformation $\mathcal{A}$ is a subset of the LVS that is shared by both the original vector v and the vector resulting from applying the linear transformation $\mathcal{A}$ to v. In other words, linear transformation $\mathcal{A}$ does not take v out of the original subspace where v originates. If that subspace has a dimension of 1 (i.e., there is only one linearly independent vector in that subspace), it is called a **1-dimensional invariant subspace**. **Eigenvector** is another common name for that lone linearly independent vector in a 1-dimensional invariant subspace. (There are many vectors in a 1-dimensional invariant subspace, but there is only *one* linearly independent one -- that comes directly from the definition of **dimension**. Basically, every vector in a 1-dimensional subspace differs from all the others only in length.)

Eigenvalue-Eigenvector equation $\mathcal{A} \cdot v = \lambda \cdot v$ $\quad v \neq 0 \quad \alpha = \text{scalar}$

A vector v that satisfies the above equation is an eigenvector. An eigenvector for linear transformation $\mathcal{A}$ is a special vector such that, when $\mathcal{A}$ is applied to it, the resulting vector is simply the original vector x multiplied by a scalar λ. The scalar λ is called the **eigenvalue**. In real Euclidean space, we can define two metrics of a vector based on the scalar product. These two metrics are: magnitude (length) and angle (direction). Thus, a linear transformation $\mathcal{A}$ applied to an eigenvector merely changes its magnitude by the associated eigenvalue λ, but not its angle. In other words, we have a case of pure expansion or contraction mapping. This characteristics makes eigenvectors an ideal candidate as the basis vectors to describe all other vectors in the LVS.

The general idea is to represent any given vector x in the LVS as a linear combination of eigenvectors v (if we can find n linearly independent ones).

$$x = \alpha_1 \cdot g_1 + \alpha_2 \cdot g_2 + \dots + \alpha_n \cdot g_n = \sum_{j=1}^{n} \alpha_j \cdot g_j$$

Given the basis g_j, a set of numbers (α_1, α_2,..., α_n) defines the vector x.

↓ Apply change-of-basis formula to rewrite x as a linear combination of eigenvectors v_j.

$$x = \beta_1 \cdot v_1 + \beta_2 \cdot v_2 + \dots + \beta_n \cdot v_n = \sum_{j=1}^{n} \beta_j \cdot v_j$$

Given the basis v_j, a set of numbers (β_1, β_2,..., β_n) defines the vector x.

If the basis v_j are orthonormal, projection gives the values of β_j.

$$\beta_j = (x, v_j)$$

Without much work, we find the effect of applying the linear transformation $\mathcal{A}$ on *any* vector x.

$$\mathcal{A}\cdot x = \beta_1\cdot(\mathcal{A}\cdot v_1) + \beta_2\cdot(\mathcal{A}\cdot v_2) + \dots + \beta_n\cdot(\mathcal{A}\cdot v_n) = \sum_{j=1}^{n} \beta_j\cdot(\mathcal{A}\cdot v_j)$$

$$\mathcal{A}\cdot x = \beta_1\cdot(\lambda_1\cdot v_1) + \beta_2\cdot(\lambda_2\cdot v_2) + \dots + \beta_n\cdot(\lambda_n\cdot v_n) = \sum_{j=1}^{n} \beta_j\cdot(\lambda_j\cdot v_j)$$

$$\mathcal{A}\cdot x = (\beta_1\cdot\lambda_1)\cdot v_1 + (\beta_2\cdot\lambda_2)\cdot v_2 + \dots + (\beta_n\cdot\lambda_n)\cdot v_n = \sum_{j=1}^{n} (\beta_j\cdot\lambda_j)\cdot v_j$$

We wish to seek out these neat vectors. An equivalent eigenvalue-eigenvector equation is,

Eigenvalue-Eigenvector equation: $(\mathcal{A} - \lambda\cdot I)\cdot v = 0$ or $(\lambda\cdot I - \mathcal{A})\cdot v = 0$ $v \neq 0$

Since v≠0, we look for solution to $|\mathcal{A} - \lambda\cdot I| = 0$ ... characteristic equation

The algebraic equation det| $\mathcal{A}$-$\lambda\cdot$I |=0 is called the **characteristic equation** for the linear transformation $\mathcal{A}$.

Note that we can determine an eigenvector only up to a scalar multiplier. If v is an eigenvector, any vector y=$\alpha\cdot$v, where $\alpha \neq 0$ is a scalar, is also en eigenvector.

$$\mathcal{A}\cdot y = \mathcal{A}\cdot(\alpha\cdot x) = \alpha\cdot(\mathcal{A}\cdot x) = \lambda\cdot(\alpha\cdot x) = \lambda\cdot y$$

In general, we would like to normalize an eigenvector so that |v|=1. Eigenvectors associated with distinct eigenvalues are linearly independent. (Why must this be true?)

Uses of eigenvalue-eigenvector: anywhere linear transformation applies. The procedure generally follows the above description. Given vectors represented with the original set of basis g, we find an equivalent representation with the eigenvectors v as the basis. Apply transformation to the eigenvectors. Express the effect of linear transformation in terms of the original set of basis.

• Example of usage: **Similarity Transform**

$$A\cdot V = V\cdot\Lambda \quad \longrightarrow \quad V^{-1}\cdot A\cdot V = \Lambda \qquad A = V\cdot\Lambda\cdot V^{-1}$$

For symmetric A $\quad V^T = V^{-1} \qquad V^T\cdot A\cdot V = \Lambda \qquad A = V\cdot\Lambda\cdot V^T$

• Example of usage: **Linear Regression**

We wish to express dependent variable as a linear combination of the independent variables X

$y = X \cdot a$ ← compare to linear transformation $\mathcal{A}$ or A → $y = \mathcal{A} \cdot x$ or $y = A \cdot x$ (just a different font)

Do not get confuse the above regression expression "y=X·a" as a linear transformation. The regression equation is merely a matrix-vector representation; X is *not* a linear transformation. Linear transformation acts on a vector in LVS and yield another vector in the same LVS. The column of numbers "a" and the dependent vector "y" in the normal equations are two very different objects!

The normal equation in linear regression is $\left(X^T \cdot X\right) \cdot a = X^T \cdot y$

Let λ and v to be the eigenvalues and eigenvectors for $X^T \cdot X$. $\left(X^T \cdot X\right) \cdot v = \lambda \cdot v$

$\left(X^T \cdot X\right) \cdot V = V \cdot \Lambda$ Because $X^T \cdot X$ is symmetrical and positive definite, $V^T = V^{-1}$

$$(X \cdot V)^T \cdot (X \cdot V) = V^T \cdot X^T \cdot X \cdot V = V^T \cdot \left(X^T \cdot X\right) \cdot V = \Lambda$$

Thus, the independent variable X expressed in terms of eigenvectors is X·V, which is called the *scores*.

The normal equation for the new independent variable X·V is $(X \cdot V)^T \cdot (X \cdot V) \cdot b = (X \cdot V)^T \cdot y$

$$\left(score^T \cdot score\right) \cdot b = score^T \cdot y$$

So, basically we express the given data X in terms of eigenvectors V; X→score, where the score vectors are mutually orthogonal (because the eigenvectors are mutually orthogonal), and the matrix of the score's scalar products is diagonal, with eigenvalues of $X^T \cdot X$ (which is the variance in the eigenvector direction) being the diagonal elements.

Find the regression coefficients b (which is reduced to scalar inverse) $\Lambda \cdot b = (X \cdot V)^T \cdot y$

After we find the regression coefficient b, the regression equation is $y = (X \cdot V) \cdot b$

• Example of usage: A set of first-order ODEs

$$\mathcal{A} \cdot x = \frac{d}{dt} x(t) = A(t) \cdot x(t)$$

$\mathcal{A}$ is a linear transformation that acts on x, a vector of functions, to yield another vector of functions. A(t) is a matrix that spells out in mathematical terms how the linear transformation $\mathcal{A}$ changes each of the x.

Let λ and v to be the eigenvalues and eigenvectors for A. $\mathcal{A} \cdot v = \lambda \cdot v$

Similarity transform $\mathcal{A} \cdot V = V \cdot J$ where J is either a diagonal or nearly diagonal Jordan matrix.

Express z in the eigenvector directions $x(t) = V \cdot z(t)$ $z(t) = V^{-1} \cdot x(t)$

The differential equation for z is $\frac{d}{dt} z(t) = V^{-1} \cdot \frac{d}{dt} x(t) = V^{-1} \cdot A(t) \cdot x(t) = V^{-1} \cdot A(t) \cdot V \cdot z(t) = J(t) \cdot z(t)$

If A(t) is a constant, $\frac{d}{dt} z(t) = J \cdot z(t)$ ⟶ $z(t) = \exp(J \cdot t) \cdot z(0)$

Find exp(J*t), which is either diagonal or in a nearly diagonal Jordan form.

Express the solution in the original vector x. $x(t) = V \cdot z(t) = V \cdot \exp(J \cdot t) \cdot z(0) = V \cdot \exp(J \cdot t) \cdot V^{-1} \cdot x(0)$

$$x(t) = \exp(A \cdot t) \cdot x(0)$$

- Example: Rotation of arrows in 3-dimension around an axis of rotation.

  The arrows in a plane to the axis of rotation resides in a 2-dimensional invariant subspace.
  The arrows // to the axis of rotation resides in a 1-dimensional invariant subspace.

- Example: 2-dimensional columns of real numbers. -- 2 eigenvectors

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \text{Define a linear transformation } \mathcal{A} \qquad \mathcal{A}\cdot x = \begin{pmatrix} 3\cdot x_1 - x_2 \\ -x_1 + 3\cdot x_2 \end{pmatrix}$$

Eigenvalue calculation:

$$\mathcal{A}\cdot v = \lambda\cdot v \qquad \mathcal{A}\cdot v = \begin{pmatrix} 3\cdot v_1 - v_2 \\ -v_1 + 3\cdot v_2 \end{pmatrix} = \begin{pmatrix} \lambda\cdot v_1 \\ \lambda\cdot v_2 \end{pmatrix}$$

$$\begin{matrix} 3\cdot v_1 - v_2 = \lambda\cdot v_1 \\ -v_1 + 3\cdot v_2 = \lambda\cdot v_2 \end{matrix} \longrightarrow \begin{matrix} (3-\lambda)\cdot v_1 - v_2 = 0 \\ -v_1 + (3-\lambda)\cdot v_2 = 0 \end{matrix} \longrightarrow \begin{matrix} \det = (3-\lambda)^2 - 1 = 0 \\ \lambda_1 = 2 \qquad \lambda_2 = 4 \end{matrix}$$

Eigenvector calculation:

$$\lambda_1 = 2 \longrightarrow \begin{matrix} v_1 - v_2 = 0 \\ -v_1 + v_2 = 0 \end{matrix}$$ The two equations are linearly dependent. $v_1 = v_2$

Thus, the eigenvector associated with λ_1=2 is $v^{<1>} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\lambda_2 = 4 \longrightarrow \begin{matrix} -v_1 - v_2 = 0 \\ -v_1 - v_2 = 0 \end{matrix}$$ The two equations are linearly dependent. $v_1 = -v_2$

Thus, the eigenvector associated with λ_2=4 is $v^{<2>} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

- Example: 2-dimensional columns of real numbers. 1 eigenvector.

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \text{Define a linear transformation A} \qquad \mathcal{A}\cdot x = \begin{pmatrix} x_1 + x_2 \\ x_2 \end{pmatrix}$$

Eigenvalue calculation:

$$\mathcal{A}\cdot v = \lambda\cdot v \qquad \mathcal{A}\cdot v = \begin{pmatrix} v_1 + v_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda\cdot v_1 \\ \lambda\cdot v_2 \end{pmatrix}$$

$$\begin{matrix} v_1 + v_2 = \lambda\cdot v_1 \\ v_2 = \lambda\cdot v_2 \end{matrix} \longrightarrow \begin{matrix} (1-\lambda)\cdot v_1 + v_2 = 0 \\ (1-\lambda)\cdot v_2 = 0 \end{matrix} \longrightarrow \begin{matrix} \det = (1-\lambda)^2 = 0 \\ \lambda_1 = 1 \qquad \lambda_2 = 1 \end{matrix}$$

Eigenvector calculation:

$$\lambda_1 = 1 \longrightarrow \begin{matrix} v_2 = 0 \\ 0 = 0 \end{matrix}$$ ← The second equation gives no new information.

Thus, the eigenvector associated with λ_1=1 is $v^{<1>} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

• Example: 2-dimensional columns of real numbers. -- 0 eigenvector

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ Define a linear transformation $\mathcal{A}$ $\mathcal{A}\cdot x = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}$

Eigenvalue calculation:

$\mathcal{A}\cdot v = \lambda\cdot v$ $\mathcal{A}\cdot v = \begin{pmatrix} v_2 \\ -v_1 \end{pmatrix} = \begin{pmatrix} \lambda\cdot v_1 \\ \lambda\cdot v_2 \end{pmatrix}$

$$\begin{matrix} v_2 = \lambda\cdot v_1 \\ -v_1 = \lambda\cdot v_2 \end{matrix} \longrightarrow \begin{matrix} -\lambda\cdot v_1 + v_2 = 0 \\ -v_1 - \lambda\cdot v_2 = 0 \end{matrix} \longrightarrow \begin{matrix} \det = \lambda^2 + 1 = 0 \\ \lambda=???? \end{matrix}$$

There is no 1-dimensional invariant subspace in this example, although there is a 2-dimensional invariant subspace. If we stick strictly with real scalars, there is no eigenvalue. If we allow complex numbers, then λ_1=i, λ_2=-i, where i is an imaginary number. Since there are no imaginary numbers in a real physical world, we will have to find a way back to the real world later if we dive into an imaginary world.

• Example: 3-dimensional columns of real numbers.

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ Define a linear transformation A $\mathcal{A}\cdot x = \begin{bmatrix} x_1 + x_2 \\ x_1 + x_3 \\ x_2 + x_3 \end{bmatrix}$

In compact matrix notation

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Eigenvalue calculation:

$$\det(A - \lambda\cdot I) = \det\left[\begin{bmatrix} 1-\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix}\right] = 0 \longrightarrow (1-\lambda)\cdot(-\lambda\cdot(1-\lambda)-1)-(1-\lambda)=0$$

$$-(\lambda-1)\cdot(\lambda-2)\cdot(\lambda+1)=0$$

$$\lambda_1 = 1 \quad \lambda_2 = -1 \quad \lambda_3 = 2$$

Eigenvector calculation:

$$\lambda_1 = 1 \longrightarrow (A - \lambda_1\cdot I)\cdot V = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\cdot\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \longrightarrow \begin{matrix} v_2 = 0 \\ v_1 - v_2 + v_3 = 0 \\ v_2 = 0 \end{matrix}$$

Only two of the above three equations are linearly dependent. $\begin{matrix} v_2 = 0 \\ v_1 = -v_3 \end{matrix}$ $v^{<1>} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Normalize $v^{<1>}$ $$v^{<1>} = \frac{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\lambda_2 = -1 \longrightarrow$ $(A - \lambda_2 \cdot I) \cdot V = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \longrightarrow$

$2 \cdot v_1 + v_2 = 0$

$v_1 + v_2 + v_3 = 0$

$v_2 + 2 \cdot v_3 = 0$

Only two of the above three equations are linearly dependent. $v_2 = -2 \cdot v_1$ $v_2 = -2 \cdot v_3$ $v^{\langle 2 \rangle} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Normalize $v^{\langle 2 \rangle}$ $v^{\langle 2 \rangle} = \dfrac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}{\left| \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right|} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$

$\lambda_3 = 2 \longrightarrow$ $(A - \lambda_3 \cdot I) \cdot V = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \longrightarrow$

$-v_1 + v_2 = 0$

$v_1 - 2 \cdot v_2 + v_3 = 0$

$v_2 - v_3 = 0$

Only two of the above three equations are linearly dependent. $v_1 = v_2$ $v_2 = v_3$ $v^{\langle 3 \rangle} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Normalize $v^{\langle 3 \rangle}$ $v^{\langle 3 \rangle} = \dfrac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{\left| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$

Combine all the eigenvalues together in a diagonal matrix. Combine all the eigenvectors together in a matrix.

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad V = \begin{pmatrix} v^{\langle 1 \rangle} & v^{\langle 2 \rangle} & v^{\langle 3 \rangle} \end{pmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Check. For orthonormal eigenvectors

$V^T \cdot V = I$ ... all permutations of scalar products

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}^T \cdot \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

... identity I ⟶ orthonormal

$V^T \cdot A \cdot V = \Lambda$

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}^T \cdot \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

... diagonal elements of eigenvalues

$A = V \cdot \Lambda \cdot V^T$

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}^T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

... what we started

- Example: 3-dimensional columns of real numbers. Repeated eigenvalues.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Eigenvalue calculation:

$$\det(A - \lambda \cdot I) = \det\left[\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}\right] = 0 \quad \longrightarrow \quad (1-\lambda)^3 = 0$$

$$\lambda_1 = 1 \qquad \lambda_2 = 1 \qquad \lambda_3 = 1$$

Eigenvector calculation:

$\lambda_1 = 1 \longrightarrow (A - \lambda_1 \cdot I) \cdot V = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \longrightarrow$ $v_2 = 0$; v_1, v_3 are any number.

eigenvector $v^{<1>} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ normalize $\longrightarrow$ $v^{<1>} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

Find $v^{<2>}$ such that it is orthogonal to the first eigenvector $v^{<1>T} \cdot v^{<2>} = 0$

$\frac{1}{\sqrt{2}} \cdot v_1 + \frac{1}{\sqrt{2}} \cdot v_3 = 0 \longrightarrow v_1 = -v_3$

eigenvector $v^{<1>} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ normalize $\longrightarrow$ $v^{<1>} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

Find $v^{<3>}$ such that it is orthogonal to the first two eigenvectors

$v^{<1>T} \cdot v^{<3>} = 0 \longrightarrow \frac{1}{\sqrt{2}} \cdot v_1 + \frac{1}{\sqrt{2}} \cdot v_3 = 0 \longrightarrow v_1 = -v_3$

$v^{<2>T} \cdot v^{<3>} = 0 \longrightarrow \frac{1}{\sqrt{2}} \cdot v_1 - \frac{1}{\sqrt{2}} \cdot v_3 = 0 \longrightarrow v_1 = v_3 \longrightarrow v_1 = v_3 = 0 \quad v_2 = 1$

$v^{<3>} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ ... This is *not* an eigenvector. It is, however, orthogonal to the first two.

Another way to find a third vector orthogonal to the first two is to rely on the *cross product* between two vectors.

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \times \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Combine all the vectors together in a matrix. This forms a good set of orthogonal basis.

$$V = \begin{pmatrix} v^{<1>} & v^{<2>} & v^{<3>} \end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Check. For orthonormal vectors

$V^T \cdot V = I$ ... scalar products

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since the last vector is not an eigenvector, the similarity transform does not leads to a diagonal matrix.

$V^T \cdot A \cdot V \neq \Lambda$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0.707 \\ 0 & 1 & 0.707 \\ 0 & 0 & 1 \end{pmatrix}$$

• Example: Consider the following linear transform that acts on real continuous functions f(t), for t=[0, 1]. (Since the dimension for this LVS is infinite, we cannot pick a finite number of basis vectors f_i and describe the action of the linear transform $\mathcal{A}$ on each of these basis vectors with a matrix A.)

$g(t) = \mathcal{A} \cdot f(t)$

Define transform $\mathcal{A}$ $\quad g(t) = \mathcal{A} \cdot f(t) = \int_0^1 f(\tau) \cdot (1 - 3 \cdot t \cdot \tau)\, d\tau$ $\quad$ cannot write as: $\mathcal{A} \cdot f(t) = A \cdot f(t)$

Check: Verify that $\mathcal{A}$ is a linear transform by checking the following two properties? (If $\mathcal{A}$ were not a linear transform, we cannot even talk about eigenvalues, and we could waste time chasing after something that does not exist.)

1. Distributive $\quad \mathcal{A} \cdot (f + g) = \mathcal{A} \cdot f + \mathcal{A} \cdot g$

$$\mathcal{A} \cdot (f + g) = \int_0^1 (f(\tau) + g(\tau)) \cdot (1 - 3 \cdot t \cdot \tau)\, d\tau = \int_0^1 f(\tau) \cdot (1 - 3 \cdot t \cdot \tau)\, d\tau + \int_0^1 g(\tau) \cdot (1 - 3 \cdot t \cdot \tau)\, d\tau$$

$$. = \mathcal{A} \cdot f + \mathcal{A} \cdot g$$

2. Associative $\quad \mathcal{A} \cdot (\alpha \cdot f) = \alpha \cdot (\mathcal{A} \cdot f)$

$$\mathcal{A} \cdot (\alpha \cdot f) = \int_0^1 (\alpha \cdot f(\tau)) \cdot (1 - 3 \cdot t \cdot \tau)\, d\tau = \alpha \cdot \int_0^1 f(\tau) \cdot (1 - 3 \cdot t \cdot \tau)\, d\tau = \alpha \cdot (\mathcal{A} \cdot f)$$

Eigenvalue calculation:

$\mathcal{A} \cdot f = \lambda \cdot f$

$$\int_0^1 f(\tau) \cdot (1 - 3 \cdot t \cdot \tau)\, d\tau = \lambda \cdot f(t)$$ ... Fredholm integral equation

Case 1. λ=0, i.e., we are looking for the null space of the linear transformation A.

$$\int_0^1 f(\tau)\,d\tau - 3\cdot t\cdot\int_0^1 f(\tau)\cdot\tau\,d\tau = \lambda\cdot f(t) = 0$$

For the above relationship to hold for all t in [0, 1], we must have

$$\int_0^1 f(\tau)\,d\tau = 0 \quad \text{and} \quad \int_0^1 f(\tau)\cdot\tau\,d\tau = 0$$

Thus, any function whose 0th moment (average) and whose 1st moment both vanish in t=[0, 1] is an eigenfunction. There are many functions that satisfies these conditions.

Case 2. λ≠0

$$\int_0^1 f(\tau)\,d\tau - 3\cdot t\cdot\int_0^1 f(\tau)\cdot\tau\,d\tau = \lambda\cdot f(t)$$

$$f(t) = \frac{1}{\lambda}\cdot\int_0^1 f(\tau)\,d\tau - \frac{3}{\lambda}\cdot\left(\int_0^1 f(\tau)\cdot\tau\,d\tau\right)\cdot t$$

Note that the integrals evaluate to scalar numbers, and the eigenfunction is a 1st degree polynomial in t.

$$f(t) = \frac{\alpha}{\lambda} - \frac{\beta}{\lambda}\cdot t$$

$$\alpha = \int_0^1 f(\tau)\,d\tau = \int_0^1 \left(\frac{\alpha}{\lambda} - \frac{\beta}{\lambda}\cdot\tau\right)d\tau = \frac{\alpha}{\lambda} - \frac{\beta}{2\cdot\lambda} \quad \longrightarrow \quad \left(1 - \frac{1}{\lambda}\right)\cdot\alpha + \frac{1}{2\cdot\lambda}\cdot\beta = 0$$

$$\beta = 3\cdot\int_0^1 f(\tau)\cdot\tau\,d\tau = 3\cdot\int_0^1 \left(\frac{\alpha}{\lambda} - \frac{\beta}{\lambda}\cdot\tau\right)\cdot\tau\,d\tau = \frac{3}{2}\cdot\frac{\alpha}{\lambda} - \frac{\beta}{\lambda} \qquad -\frac{3}{2\cdot\lambda}\cdot\alpha + \left(1 + \frac{1}{\lambda}\right)\cdot\beta = 0$$

Set det|...|=0 (Note that we continue to have a characteristic equation det|...|=0 that governs the eigenvalues even when we cannot describe the action of $\mathcal{A}$ as "g=A·f" -- remember, the number of basis vectors is infinite.)

$$\left(1 - \frac{1}{\lambda}\right)\cdot\left(1 + \frac{1}{\lambda}\right) + \frac{1}{2\cdot\lambda}\cdot\frac{3}{2\cdot\lambda} = 0$$ ... characteristic equation for the linear transformation $\mathcal{A}$

$$\lambda_1 = \frac{1}{2} \qquad \lambda_2 = -\frac{1}{2}$$

Eigenvector calculation for $\lambda_1 = \frac{1}{2}$

$$(1 - 2)\cdot\alpha + \frac{2}{2}\cdot\beta = 0$$

$$-3\cdot\alpha + (1 + 2)\cdot\beta = 0$$

Only one of the above two equations are linearly independent. $\alpha = \beta$

$f(t) = 2\cdot\alpha\cdot(1 - t)$ is an eigenfunction. Since any scalar multiple of this function is also an eigenfunction, we can take out the constant factor, or we can normalize the eigenfunction.

$$f_1(t) = 1 - t$$

Eigenvector calculation for $\lambda_1 = -\frac{1}{2}$

$$(1+2)\cdot\alpha - \frac{2}{2}\cdot\beta = 0$$

$$3\cdot\alpha + (1-2)\cdot\beta = 0$$

Only one of the above two equations are linearly independent. $3\cdot\alpha = \beta$

$f(t) = 2\cdot\alpha\cdot(1-t)$ is an eigenfunction. We can take out the constant factor, or we can normalize the eigenfunction.

$f_2(t) = 1 - 3\cdot t$

Orthogonality between various eigenfunctions $f_0(t)$ for $\lambda_0=0$, $f_1(t)$ for $\lambda_1=1/2$, and $f_2(t)=-1/2$.

Define a scalar product. $(f,g) = \int_0^1 f(\tau)\cdot g(\tau)\,d\tau$

$$\left(f_0, f_1\right) = \int_0^1 f_0(\tau)\cdot f_1(\tau)\,d\tau = \int_0^1 f_0(\tau)\cdot(1-\tau)\,d\tau = \int_0^1 f_0(\tau)\,d\tau - \int_0^1 f_0(\tau)\cdot\tau\,d\tau = 0$$

$$\left(f_0, f_2\right) = \int_0^1 f_0(\tau)\cdot f_2(\tau)\,d\tau = \int_0^1 f_0(\tau)\cdot(1-3\cdot\tau)\,d\tau = \int_0^1 f_0(\tau)\,d\tau - 3\cdot\int_0^1 f_0(\tau)\cdot\tau\,d\tau = 0$$

$$\left(f_1, f_2\right) = \int_0^1 f_1(\tau)\cdot f_2(\tau)\,d\tau = \int_0^1 (1-\tau)\cdot(1-3\cdot\tau)\,d\tau = 0$$

In summary, we have $\left(f_0, f_1\right) = \left(f_0, f_2\right) = \left(f_1, f_2\right) = 0$ Thus, all three eigenfunctions are mutually orthogonal.

- Example: Successive approximation on a vector.

$$x = \phi(x)$$

If ϕ is a linear transformation, then finding solution x is identical to finding an eigenvector that corresponds to $\lambda=1$.

$\phi(x) = \lambda\cdot x \longrightarrow \lambda = 1$ This is a bit silly, but it is sometimes easier to see things, after we arrange them in the "standard" form $\mathcal{A}\cdot x = x$.