Contents

Introduction

I had three reasons for compiling this solution set. First, I was inspired by the works of the great Homer Reid, whos website link dons my own. Second, I thought that a "database" of solutions to several problems would be helpful to myself in the future. Third, as the graduate experience in the Physics department at the University of Maryland is a difficult one, and as my generation of incoming students found resources of graduate level physics solutions to be substantially less than those of under-graduate level, I thought I would provide a resource (however meager) to those that enter the program after me.

I am aware of the potential for this resource to be abused. However, I believe that one could not have become a graduate student in physics by simply copying solutions and therefore I have every confidence that one would not begin such practices upon entering a graduate program. I entered the field of physics because I have a deep curiosity for how the universe works and a stubborn desire to understand it; as a result, I have found it difficult to "copy" solutions, regardless of how accessible they may be. Indeed, I have found that after attempting to solve a problem unsuccessfully, I may spend at least as much time studying someone else's solution, determined to understand it as best as I could. I am sure that those who find this compilation of solutions helpful also share these sentiments concerning its use and abuse.

As a formality, I warn the reader that these solutions are not guaranteed to be correct, accurate, or complete. The solutions, as presented, are not necessarily deserving of "full credit". That being written, I have tried to make corrections from the original solutions that I submitted to the instructor. If you find any errors, please contact me using the link found on my website's Solutions page.

Chapter 5

5.32)

Eigenstates of $A\mathbf{S}_1 \cdot \mathbf{S}_2$ are $|\,11\rangle$, $|\,10\rangle$, $|\,1-1\rangle$, and $|\,00\rangle$ where

$$\begin{aligned} |\,11\rangle &= |\,++\rangle \\ |\,10\rangle &= \frac{1}{\sqrt{2}}(|\,+-\rangle + |\,-+\rangle) \\ |\,1-1\rangle &= \frac{1}{\sqrt{2}}(|\,+-\rangle - |\,-+\rangle) \\ |\,00\rangle &= |\,--\rangle \end{aligned}$$

Taking $\mathbf{H}_0 = A\mathbf{S}_1 \cdot \mathbf{S}_2$ and $\mathbf{V} = \frac{eB}{m_e c}(S_{1z} - S_{2z})$ we have matrices in the $|\,11\rangle$, $|\,1-1\rangle$, $|\,10\rangle$, and $|\,00\rangle$ basis

$$\mathbf{H}_0 = \frac{A\hbar^2}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \tag{5.32.1}$$

$$\mathbf{V} = \frac{eB}{m_e c}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \hbar \\ 0 & 0 & \hbar & 0 \end{bmatrix} \tag{5.32.2}$$

Since $\mathbf{V}$ only couples two non-degenerate states, we can use non-degenerate perturbation theory.

$$\begin{aligned} \Delta E_{10}^{(1)} &= 0 \\ \Delta E_{10}^{(2)} &= \frac{|\,\langle 00\,|V|\,10\,\rangle\,|^2}{(1+3)\,A\hbar^2/4} \\ &= \frac{1}{A}\left(\frac{eB}{m_e c}\right)^2 \end{aligned} \tag{5.32.3}$$

$$\Delta E_{00}^{(1)} = 0 \tag{5.32.4}$$

$$\Delta E_{00}^{(2)} = \frac{|\,\langle 10\,|V|\,00\,\rangle\,|^2}{(-3-1)\,A\hbar^2/4} \tag{5.32.5}$$

$$= -\frac{1}{A}\left(\frac{eB}{m_e c}\right)^2 \tag{5.32.6}$$

$$\Delta E_{11} = \Delta E_{1-1} = 0 \tag{5.32.7}$$

$$E_{11} = E_{1-1} = \frac{A\hbar^2}{4} \tag{5.32.8}$$

$$E_{10} = \frac{A\hbar^2}{4} + \frac{1}{A}\left(\frac{eB}{m_e c}\right)^2 \tag{5.32.9}$$
$$E_{00} = -\frac{3A\hbar^2}{4} - \frac{1}{A}\left(\frac{eB}{m_e c}\right)^2 \tag{5.32.10}$$

This is consistent with the exact expression given in the text, provided $eB/m_e c\hbar A \ll 1$.

If we align the oscillating $\mathbf{B}$ field in either the x or y direction, we will introduce tensors of rank one, thereby introducing a selection rule $\Delta m = 1$. Since $\mathbf{V}$ only couples states with $m = 0$, we must have a selection rule $\Delta m = 0$; this can only be accomplished by using an operator of rank zero, therefore $\mathbf{B}$ must be in the z direction.

The eigenvectors are

$$|\,11\rangle = |\,11\rangle \tag{5.32.11}$$
$$|\,1-1\rangle = |\,1-1\rangle \tag{5.32.12}$$
$$\begin{aligned} |\,10\rangle &= |\,10\rangle^{(0)} + \frac{\langle 00\,|V|\,10\,\rangle}{(1+3)\,A\hbar^2/4}|\,00\rangle^{(0)} \\ &= |\,10\rangle^{(0)} + \frac{eB}{m_e c}\frac{1}{A\hbar}|\,00\rangle^{(0)} \end{aligned} \tag{5.32.13}$$
$$\begin{aligned} |\,00\rangle &= |\,00\rangle^{(0)} + \frac{\langle 10\,|V|\,00\,\rangle}{(-3-1)\,A\hbar^2/4}|\,10\rangle^{(0)} \\ &= |\,00\rangle^{(0)} - \frac{eB}{m_e c}\frac{1}{A\hbar}|\,10\rangle^{(0)} \end{aligned} \tag{5.32.14}$$

If the $\mathbf{B}(t) = B_0 \cos\omega t\hat{\mathbf{z}}$ and $\mathbf{B}(0) = |\,10\rangle$, then we can use time-dependent perturbation theory:

$$\begin{aligned} c_{00}^{(1)} &= -\frac{i}{\hbar}\int_0^t e^{i\omega_{fi}t'}\,\langle 00\,|V|\,10\,\rangle\;dt' \\ &= -i\frac{eB_0}{m_e c}\int_0^t e^{i\omega_{fi}t'}\cos\omega t'\;dt' \end{aligned}$$

Taking $\omega_{fi} = \omega$ as the "correct" frequency, the integral becomes

$$c_{00}^{(1)} = \frac{t}{2} + \frac{i}{4\omega_{fi}}\left(e^{-2i\omega_{fi}t} - 1\right) \tag{5.32.15}$$

Substituting $\omega_{fi} = (E_{00} - E_{10})/\hbar = -A\hbar$, we have

$$\begin{aligned} |\,\psi(t)\,\rangle &= |\,10\rangle + \left(\frac{t}{2} - \frac{i}{4A\hbar}\left(e^{i2A\hbar t} - 1\right)\right)|\,00\rangle \\ &= |\,10\rangle + \left(\frac{t}{2} + \frac{e^{iA\hbar t}}{2A\hbar}\sin A\hbar t\right)|\,00\rangle \end{aligned} \tag{5.32.16}$$

5.33)

If we begin with Eqn. (5.7.8) from the text we can write

$$w_{ni} = \frac{2\pi}{\hbar}\frac{e^2}{m_e^2c^2}\,|A_0|^2 \sum_\lambda \left|\langle\, n\,|e^{i(\omega/c)(\hat{\mathbf{n}}\cdot\mathbf{r})}\hat{\epsilon}_\lambda\cdot\mathbf{p}|\, i\,\rangle\right|^2 \delta(E_n - E_i - \hbar\omega) \tag{5.33.1}$$

where λ indicates polarization directions for the photon. We note that we will only get angular dependence from the $|V_{ni}|^2$ term. Ignoring the multiplicative coefficients, the angular dependence is proportional to

$$\sum_\lambda \left|\langle\, n\,|e^{i(\omega/c)(\hat{\mathbf{n}}\cdot\mathbf{r})}\hat{\epsilon}_\lambda\cdot\mathbf{p}|\, i\,\rangle\right|^2 \tag{5.33.2}$$

If we take $e^{i(\omega/c)(\hat{\mathbf{n}}\cdot\mathbf{r})} \cong 1$, the $E1$ approximation, then we have

$$\begin{aligned}\sum_\lambda |\langle\, n\,|\hat{\epsilon}_\lambda\cdot\mathbf{p}|\, i\,\rangle|^2 &\propto \sum_\lambda |\langle\, n\,|\,[\mathbf{H},\hat{\epsilon}_\lambda\cdot\mathbf{r}]\,|\, i\,\rangle|^2 \\ &\propto \sum_\lambda |\hat{\epsilon}_\lambda\cdot\langle\, n\,|\mathbf{r}|\, i\,\rangle|^2\end{aligned}$$

We note that

$$\begin{aligned}\mathbf{r} &= x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + y\hat{\mathbf{y}} \\ &= A_1Y_1^1\hat{\mathbf{e}}_- + A_{-1}Y_1^{-1}\hat{\mathbf{e}}_+ + A_0Y_0^0\hat{\mathbf{e}}_\mathbf{0}\end{aligned} \tag{5.33.3}$$

Where A_α are constants of proportionality, and $\hat{\mathbf{e}}_+ \sim \hat{\mathbf{x}} + i\hat{\mathbf{y}}$, etc. We are given in the problem that $\Delta m = -1$ or $m' - m = -1$, therefore we must take only the Y_1^1 term from $\mathbf{r}$. We now have

$$\begin{aligned}\sum_\lambda |\langle\, n\,|\hat{\epsilon}_\lambda\cdot\mathbf{p}|\, i\,\rangle|^2 &\propto \sum_\lambda \left|\hat{\epsilon}_\lambda\cdot\hat{\mathbf{e}}_+\,\langle\, m-1\,|Y_1^1|\, m\,\rangle\right|^2 \\ &\propto \sum_\lambda |\hat{\epsilon}_\lambda\cdot\hat{\mathbf{e}}_+|^2 = \sum_\lambda |\hat{\epsilon}_\lambda\cdot\hat{\mathbf{e}}_+|^2\end{aligned}$$

If we choose our polarization vectors to be (where $\hat{\mathbf{k}}$ is the direction of the photon)

$$\hat{\epsilon}_1 = \hat{\mathbf{k}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \tag{5.33.4}$$

$$\hat{\epsilon}_2 = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \tag{5.33.5}$$

$$\hat{\epsilon}_3 = \hat{\epsilon}_1\times\hat{\epsilon}_2 = -\cos\theta\cos\phi\,\hat{\mathbf{x}} - \cos\theta\sin\phi\,\hat{\mathbf{y}}\sin\theta\,\hat{\mathbf{z}} \tag{5.33.6}$$

then we only need to sum over λ from 2 to 3. Now the angular dependence is proportional to

$$\begin{aligned}\sum_{\lambda=2}^{3} |\hat{\epsilon}_\lambda\cdot\hat{\mathbf{e}}_+|^2 &\propto |\hat{\epsilon}_2\cdot(\hat{\mathbf{x}}+i\hat{\mathbf{y}})|^2 + |\hat{\epsilon}_3\cdot(\hat{\mathbf{x}}+i\hat{\mathbf{y}})|^2 \\ &\propto \sin^2\phi + \cos^2\phi + \cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi \\ &\propto 1 + \cos^2\theta\end{aligned} \tag{5.33.7}$$

So the angular dependence goes as $1 + \cos^2\theta$, ignoring factors that are independent of θ.

5.34)

For both $t < 0$ and $t > 0$, we have hydrogenic wave functions. The only difference between the two wave functions is the magnitude of the charge that shows up in the radial part of the wave function. Therefore, any angular integrals between ground states will evaluate to one and we only need to consider the radial integrals. Taking $|\,\rangle^-$ for $t < 0$ and $|\,\rangle^+$ for $t > 0$, we can write

$$\begin{aligned}
\psi_{10}^{+} &= \frac{1}{\sqrt{\pi}}\left(\frac{2}{a_0}\right)^{3/2} e^{-2r/a_0} \\
\psi_{10}^{-} &= \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0} \\
{}^{+}\langle\, 0 \,|\, 0 \,\rangle^{-} &= \int_0^{2\pi}\int_0^{\infty} \psi_{10}^{+*}\psi_{10}^{-} r^2 \, dr \, d\Omega \\
&= \frac{4\pi}{\pi}\frac{\sqrt{8}}{a_0^3}\int_0^{\infty} e^{-3r/a_0} r^2 \, dr \\
&= \frac{4\sqrt{8}}{a_0^3}\frac{2!a_0^3}{3^3} \\
&= \frac{16\sqrt{2}}{27}
\end{aligned} \tag{5.34.1}$$

The probability of being in the ground state of helium is then

$$\begin{aligned}
|{}^{+}\langle\, 0 \,|\, 0 \,\rangle^{-}|^2 &= \left(\frac{16\sqrt{2}}{27}\right)^2 \\
&\approx 0.702 = 70.2\%
\end{aligned} \tag{5.34.2}$$

5.35)

Using Eqn. (5.6.34) from the text we may write

$$w_{fi} = \left(\frac{2\pi}{\hbar}\right)|V_{fi}|^2 \rho_f(E) \tag{5.35.1}$$

Taking the density of states to be (see page 340)

$$\rho_f(E) = \left(\frac{L}{2\pi}\right)^3 \frac{m_e k_f}{\hbar^2} d\Omega \tag{5.35.2}$$

we now have

$$w_{fi} = \left(\frac{2\pi}{\hbar}\right)|V_{fi}|^2 \left(\frac{L}{2\pi}\right)^3 \frac{m_e k_f}{\hbar^2} d\Omega \tag{5.35.3}$$

We can compute the matrix element, V_{fi} as follows:

$$V_{fi} = \int_V \langle\, \mathbf{k}_f \,|\frac{V_0}{2}\left(e^{i(kz-\omega t)} + e^{-i(kz-\omega t)}\right)|\, i\rangle \; d\tau$$

Since we are only interested in the absorbtion term, we will ignore the $+i\omega t$ term. Using

$$|\, i\,\rangle = \psi_{10} = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} \quad \text{and} \tag{5.35.4}$$

$$\langle \mathbf{r}\,|\,\mathbf{k}_f\,\rangle = \frac{1}{L^{3/2}} e^{i\mathbf{k}_f\cdot\mathbf{r}} \tag{5.35.5}$$

we now have

$$\begin{aligned} V_{fi} &= \int_V \langle\, \mathbf{k}_f\,| \frac{V_0}{2}\left(e^{i(kz-\omega t)}\right) |\, i\,\rangle \; d\tau \\ &= \frac{V_0}{2L^{3/2}} \frac{1}{\sqrt{\pi a_0^3}} \int_V e^{-i\mathbf{k}_f\cdot\mathbf{r}+ikz-i\omega t-r/a_0} \; d\tau \\ &= \frac{V_0}{2L^{3/2}} \frac{e^{-i\omega t}}{\sqrt{\pi a_0^3}} \int_V e^{-i\mathbf{k}_f\cdot\mathbf{r}+ikz-r/a_0} \; d\tau \end{aligned} \tag{5.35.6}$$

Let $\mathbf{q} = \mathbf{k}_f - k\hat{\mathbf{z}}$, then

$$\begin{aligned} V_{fi} &= \frac{V_0}{2L^{3/2}} \frac{e^{-i\omega t}}{\sqrt{\pi a_0^3}} \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} e^{-i\mathbf{q}_f\cdot\mathbf{r}-r/a_0} r^2 \sin\theta \; d\theta dr d\phi \\ &= \frac{V_0\pi}{L^{3/2}} \frac{e^{-i\omega t}}{\sqrt{\pi a_0^3}} \int_0^{\infty} e^{-r/a_0}\; r^2 dr \; \int_{-1}^{1} e^{-iqr\cos\theta} \; d(\cos\theta) \\ &= -\frac{V_0\pi}{L^{3/2}} \frac{e^{-i\omega t}}{\sqrt{\pi a_0^3}} \frac{1}{iq} \int_0^{\infty} e^{-r/a_0}\left(e^{iqr} - e^{-iqr}\right) r \; dr \\ &= -\frac{V_0\pi}{L^{3/2}} \frac{e^{-i\omega t}}{\sqrt{\pi a_0^3}} \frac{1}{iq}\left(\frac{1}{(-1/a_0+iq)^2} - \frac{1}{(-1/a_0-iq)^2}\right) \\ &= -\frac{V_0\pi}{L^{3/2}} \frac{e^{-i\omega t}}{\sqrt{\pi a_0^3}} \frac{4/a_0}{(1/a_0^2+q^2)^2} \end{aligned} \tag{5.35.7}$$

Substituting into Eqn. (21), we get for the angular dependence

$$\frac{w_{fi}}{d\Omega} = \frac{2\pi}{\hbar}\frac{V_0^2}{a_0^5}\frac{16\pi}{(2\pi)^3}\frac{m_e k_f}{\hbar^2}\frac{1}{\left[(1/a_0^2+q^2)\right]^4} \tag{5.35.8}$$

where

$$q^2 = k_f^2 + k^2 - k_f k \cos\theta \tag{5.35.9}$$

This system is completely analogous to the photo-electric effect, except in this situation we have no $\hat{\epsilon}\cdot\mathbf{k}_f$ term present. Comparing our result with that from the text, we see the same angular dependence if one removes the $(\hat{\epsilon}\cdot\mathbf{k}_f)^2$ term from Eqn. (5.7.36).

5.37)

The energy for a particle in a one-dimensional box is

$$\begin{aligned} E_n &= \frac{\hbar^2\pi^2 n^2}{2mL^2} \qquad (5.37.1)\\ \Delta E &= \frac{\hbar^2\pi^2((n+1)^2-n^2)}{2mL^2} \\ &= \frac{\hbar^2\pi^2(2n+1)}{2mL^2} \end{aligned}$$

If E_n is taken to be very large, then $n \gg 1$ and we have

$$\Delta E = \frac{\hbar^2\pi^2 n}{mL^2}$$

Now using

$$n = \sqrt{\frac{2mL^2}{\hbar^2\pi^2}E}$$

we have

$$\begin{aligned} \rho_n(E) &= \frac{\Delta n}{\Delta E} = \frac{1}{\Delta E} = \frac{mL^2}{\hbar^2\pi^2}\sqrt{\frac{\hbar^2\pi^2}{2mL^2E}} \\ &= \frac{L}{\pi\hbar}\sqrt{\frac{m}{2E}} \qquad (5.37.2) \end{aligned}$$

5.38)

This problem is identical to the photo-electric effect, so that we may use Eqn. (5.7.32) from the text.

$$\frac{d\sigma}{d\Omega} = \frac{4\pi^2\alpha\hbar}{m_e^2\omega}\left|\langle \mathbf{k}_f \,| e^{i(\omega/c)(\hat{\mathbf{n}}\cdot\mathbf{r})}\hat{\epsilon}\cdot\mathbf{p}|\, i\,\rangle\right|^2 \frac{m_e k_f V}{\hbar^2(2\pi)^3} \qquad (5.38.1)$$

For a three-dimensional isotropic harmonic oscillator initially in the ground state, the wave functions for $|\,\mathbf{k}\rangle$ and $|\,0\rangle$ are

$$\begin{aligned} \langle \mathbf{k}\,|\,\mathbf{r}\,\rangle &= \frac{1}{\sqrt{V}}e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} \qquad (5.38.2)\\ \langle 0\,|\,\mathbf{r}\,\rangle &= \left(\frac{m_e\omega_0}{\pi\hbar}\right)^{3/4} e^{-\xi^2/2} \qquad (5.38.3) \end{aligned}$$

where $\xi^2 = A^2(x^2+y^2+z^2)$ and $A^2 = m_e\omega_0/\hbar$, given by Eqns. (A.4.2) and (A.4.3). Inserting into Eqn. (1) and taking $\mathbf{q} = \mathbf{p}/\hbar - (\omega/c)\hat{\mathbf{k}}$ now gives us

$$\frac{d\sigma}{d\Omega} = \frac{4\pi^2\alpha\hbar}{m_e^2\omega}\frac{m_e kV}{\hbar^2(2\pi)^3}\left|\int \frac{1}{\sqrt{V}}e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}e^{i(\omega/c)(\hat{\mathbf{k}}\cdot\mathbf{r})}\hat{\epsilon}\cdot\mathbf{p}\left(\frac{m_e\omega_0}{\pi\hbar}\right)^{3/4} e^{-\xi^2/2}\, d^3\mathbf{r}\right|^2$$

$$
\begin{aligned}
&= \frac{\alpha}{m_e\omega}\frac{k|-i\hbar|^2}{\hbar(2\pi)}\left(\frac{m_e\omega_0}{\pi\hbar}\right)^{3/2}\left|\hat{\epsilon}\cdot\int e^{-i\mathbf{q}\cdot\mathbf{r}}\nabla e^{-\xi^2/2}\ d^3\mathbf{r}\right|^2 \\
&= \frac{\alpha}{m_e\omega}\frac{\hbar k}{2\pi}\left(\frac{m_e\omega_0}{\pi\hbar}\right)^{3/2}\left|\hat{\epsilon}\cdot\left[\hat{\mathbf{r}}\ e^{-i\mathbf{q}\cdot\mathbf{r}}e^{-\xi^2/2}+\int i\mathbf{q}e^{-i\mathbf{q}\cdot\mathbf{r}}e^{-\xi^2/2}\ d^3\mathbf{r}\right]\right|^2 \\
&= \frac{\alpha}{m_e\omega}\frac{\hbar k}{2\pi}\left(\frac{m_e\omega_0}{\pi\hbar}\right)^{3/2}\left|\hat{\epsilon}\cdot\mathbf{q}\int e^{-A^2/2(x^2+i2q_xx/A^2)}e^{-A^2/2(y^2+i2q_yy/A^2)}e^{-A^2/2(z^2+i2q_zz/A^2)}\ d^3\mathbf{r}\right|^2 \\
&= \frac{\alpha}{m_e\omega}\frac{\hbar k}{2\pi}\left(\frac{m_e\omega_0}{\pi\hbar}\right)^{3/2}\left|\hat{\epsilon}\cdot\mathbf{q}\ e^{-q^2/2A^4}\int e^{-A^2/2(x+iq_x/A^2)^2}e^{-A^2/2(y+iq_y/A^2)^2}e^{-A^2/2(z+iq_z/A^2)^2}\ d^3\mathbf{r}\right|^2 \\
&= \frac{\alpha}{m_e\omega}\frac{\hbar k}{2\pi}\left(\frac{m_e\omega_0}{\pi\hbar}\right)^{3/2}\left|\hat{\epsilon}\cdot\mathbf{q}\ e^{-q^2/2A^4}\left(\frac{\sqrt{2\pi}}{A}\right)^3\right|^2
\end{aligned}
$$

$$
\frac{d\sigma}{d\Omega} = \frac{4\alpha\hbar^2k^3}{m_e^2\omega\omega_0}\sqrt{\frac{\pi\hbar}{m_e\omega_0}}\exp\left\{-\frac{\hbar}{m_e\omega_0}\left[k^2+\left(\frac{\omega}{c}\right)^2\right]\right\}\sin^2\theta\cos^2\phi\exp\left\{\left(\frac{2\hbar k\omega}{m_e\omega_0c}\right)\cos\theta\right\} \tag{5.38.4}
$$

5.39)

We can transform our wave function from position space to momentum space using the following equations:

$$
\psi(\mathbf{r}) = \langle\,\mathbf{r}\,|\,\psi\,\rangle = \frac{1}{\sqrt{\pi}}\frac{1}{a_0^{3/2}}e^{-r/a_0} \tag{5.39.1}
$$

$$
\phi(\mathbf{p}) = \langle\,\mathbf{p}\,|\,\psi\,\rangle = \left(\frac{1}{2\pi\hbar}\right)^{3/2}\int e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}\psi(\mathbf{r})\ d^3\mathbf{r} \tag{5.39.2}
$$

Eqn. (2) is just a Fourier transform of the position-space wave function. We can carry out the integral as follows.

$$
\begin{aligned}
\phi(\mathbf{p}) &= \left(\frac{1}{2\pi\hbar}\right)^{3/2}\frac{1}{\sqrt{\pi}}\frac{1}{a_0^{3/2}}\int e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}e^{-r/a_0}\ d^3\mathbf{r} \\
&= \left(\frac{1}{2\hbar a_0}\right)^{3/2}\frac{2}{\pi^2}\int_0^\infty\int_{-1}^1 e^{-ipr\cos\theta/\hbar}e^{-r/a_0}r^2\ dr\ d(\cos\theta) \\
&= \left(\frac{1}{2\hbar a_0}\right)^{3/2}\frac{2}{\pi^2}\frac{-i\hbar}{p}\int_0^\infty(e^{ipr/\hbar}-e^{-ipr/\hbar})e^{-r/a_0}r\ dr \\
&= \left(\frac{1}{2\hbar a_0}\right)^{3/2}\frac{4}{\pi^2}\frac{\hbar}{p}\int_0^\infty\sin\left(\frac{pr}{\hbar}\right)e^{-r/a_0}r\ dr \\
&= \left(\frac{1}{2\hbar a_0}\right)^{3/2}\frac{4}{\pi^2}\frac{\hbar}{p}\frac{2a_0^3\hbar^3p}{(\hbar^2+a_0^2p^2)^2} = \frac{\hbar\,(2\hbar a_0)^{3/2}}{\pi^2}\frac{1}{(\hbar^2+a_0^2p^2)^2}
\end{aligned} \tag{5.39.3}
$$

We can now obtain the probability for $|\phi(\mathbf{p})|^2d^3p$.

$$
|\phi(\mathbf{p})|^2d^3p = \frac{\hbar^2\,(2\hbar a_0)^3}{\pi^4}\frac{1}{(\hbar^2+a_0^2p^2)^4}\ d^3p
$$

$$= \frac{8\hbar^5 a_0^3}{\pi^4} \frac{1}{(\hbar^2 + a_0^2 p^2)^4}\, d^3p \tag{5.39.4}$$

5.40)

The decay time, τ, is the inverse of the transition rate, w_{fi}. We use the following expressions.

$$w_{fi} = \frac{1}{3}\sum_m \int_\Omega \frac{2\pi}{\hbar}\sum_\lambda \left|\langle\, 2\mathrm{p_m}\,|\frac{-eA_0}{m_e c}\hat{\epsilon}_\lambda\cdot\mathbf{p}|\,1\mathrm{s_0}\,\rangle\right|^2 \rho_{photon} = \frac{1}{\tau} \tag{5.40.1}$$

$$\rho_{photon}(\omega) = \frac{Vd^3\mathbf{k}}{(2\pi)^3} = \frac{V\omega^2 d\Omega}{(2\pi)^3\hbar c^3} \tag{5.40.2}$$

$$\mathbf{p} = \frac{m_e}{i\hbar}[\mathbf{r},\mathbf{H}] \tag{5.40.3}$$

$$E_n = -\frac{e^2}{2a_0 n^2} \tag{5.40.4}$$

$$A_0^2 = \frac{2\pi\hbar c^2}{V\omega} \tag{5.40.5}$$

We can now substitute Eqns. (3) and (2) into (1). The commutation relation involving **H** will introduce a factor that is the difference between the $2p$ and $1s$ energy levels.

$$\begin{aligned} w_{fi} &= \frac{1}{3}\sum_m \int_\Omega \frac{2\pi}{\hbar}\sum_\lambda \left|\langle\, 2\mathrm{p_m}\,|\frac{-eA_0}{i\hbar c}\hat{\epsilon}_\lambda\cdot[\mathbf{r},\mathbf{H}]\,|\,1\mathrm{s_0}\,\rangle\right|^2 \frac{V\omega^2}{(2\pi)^3\hbar c^3}d\Omega \\ &= \frac{1}{3}\frac{2\pi}{\hbar}\frac{e^2A_0^2}{\hbar^2c^2}\frac{V\omega^2}{(2\pi)^3\hbar c^3}(E_1-E_2)^2\int_\Omega\sum_m\sum_\lambda |\hat{\epsilon}_\lambda\cdot\langle\, 2\mathrm{p_m}\,|\mathbf{r}|\,1\mathrm{s_0}\,\rangle|^2\, d\Omega \end{aligned} \tag{5.40.6}$$

Now $\mathbf{r}$ is a combination of radial wave-functions and spherical harmonics. Let us first evaluate the matrix element, $\langle\, 2\mathrm{p_m}\,|\mathbf{r}|\,1\mathrm{s_0}\,\rangle$.

$$\begin{aligned} \langle\, 2\mathrm{p_m}\,|\mathbf{r}|\,1\mathrm{s_0}\,\rangle &= \int R_{21}^* R_{10}\, r^3\, dr \int_\Omega Y_1^{m*}\sqrt{\frac{4\pi}{3}}\left(-Y_1^1\hat{e}_- - Y_1^{-1}\hat{e}_+ + Y_1^0\hat{e}_0\right) Y_0^0\, d\Omega \\ &= \frac{1}{2^{3/2}a_0^4}\frac{2}{3}\int e^{-3r/2a_0}\, r^4\, dr \int_\Omega Y_1^{m*}\left(-Y_1^1\hat{e}_- - Y_1^{-1}\hat{e}_+ + Y_1^0\hat{e}_0\right) d\Omega \\ &= \frac{1}{2^{1/2}a_0^4}\frac{1}{3}\cdot\frac{4!}{(3/2a_0)^5}\cdot(-1)^m\hat{e}_{-m} \\ &= \frac{2^{15/2}a_0}{3^5}\cdot(-1)^m\hat{e}_{-m} \end{aligned} \tag{5.40.7}$$

Substituting back into Eqn. (6) then gives us

$$w_{fi} = \frac{1}{3}\frac{2\pi}{\hbar}\frac{e^2A_0^2}{\hbar^2c^2}\frac{V\omega^2}{(2\pi)^3\hbar c^3}(E_1-E_2)^2\frac{2^{15}a_0^2}{3^{10}}\int_\Omega\sum_m\sum_\lambda|\hat{\epsilon}_\lambda\cdot\hat{e}_{-m}|^2\, d\Omega$$

$$
\begin{aligned}
&= \frac{e^2 A_0^2}{\hbar^4 c^5} \frac{V\omega^2}{(2\pi)^2} \frac{3^2 e^4}{2^6 a_0^2} \frac{2^{15} a_0^2}{3^{11}} \int_\Omega \sum_m \sum_\lambda \left|\hat{\epsilon}_\lambda \cdot \hat{e}_{-m}\right|^2 d\Omega \\
&= \frac{A_0^2}{\hbar^4 c^5} \frac{V\omega^2 e^6}{(2\pi)^2} \frac{2^9}{3^9} \int_\Omega \sum_m \sum_\lambda \left|\hat{\epsilon}_\lambda \cdot \hat{e}_{-m}\right|^2 d\Omega \\
&= \frac{1}{\hbar^3 c^3} \frac{\omega e^6}{2\pi} \frac{2^9}{3^9} \int_\Omega \sum_m \sum_\lambda \left|\hat{\epsilon}_\lambda \cdot \hat{e}_{-m}\right|^2 d\Omega \qquad (5.40.8)
\end{aligned}
$$

Since we can rotate the electron propogation vector about the z axis without altering the spherical harmonic calculations that we already performed, we can choose the coordinate system such that $\hat{\mathbf{k}}$ is in the yz plane. We can therefore choose three ortho-normal polarization vectors: $\hat{e}_\lambda = \{\hat{\mathbf{x}}, \hat{\mathbf{k}}, -\hat{\mathbf{y}}\cos\theta + \hat{\mathbf{z}}\sin\theta\}$. We now only have to sum over the first and last polarization directions:

$$
\begin{aligned}
\sum_m \sum_\lambda \left|\hat{\epsilon}_\lambda \cdot \hat{e}_{-m}\right|^2 &= \sum_m \left[\left|\hat{\mathbf{x}} \cdot \hat{e}_{-m}\right|^2 + \left|(-\hat{\mathbf{y}}\cos\theta + \hat{\mathbf{z}}\sin\theta) \cdot \hat{e}_{-m}\right|^2\right] \\
&= \left|\hat{\mathbf{x}} \cdot \hat{e}_{-1}\right|^2 + \left|(-\hat{\mathbf{y}}\cos\theta + \hat{\mathbf{z}}\sin\theta) \cdot \hat{e}_{-1}\right|^2 \\
&\quad + \left|\hat{\mathbf{x}} \cdot \hat{e}_{0}\right|^2 + \left|(-\hat{\mathbf{y}}\cos\theta + \hat{\mathbf{z}}\sin\theta) \cdot \hat{e}_{0}\right|^2 \\
&\quad + \left|\hat{\mathbf{x}} \cdot \hat{e}_{1}\right|^2 + \left|(-\hat{\mathbf{y}}\cos\theta + \hat{\mathbf{z}}\sin\theta) \cdot \hat{e}_{1}\right|^2 \\
&= \left|\hat{\mathbf{x}} \cdot \frac{\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}}\right|^2 + \left|(-\hat{\mathbf{y}}\cos\theta + \hat{\mathbf{z}}\sin\theta) \cdot \frac{\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}}\right|^2 \\
&\quad + \left|\hat{\mathbf{x}} \cdot \hat{\mathbf{z}}\right|^2 + \left|(-\hat{\mathbf{y}}\cos\theta + \hat{\mathbf{z}}\sin\theta) \cdot \hat{\mathbf{z}}\right|^2 \\
&\quad + \left|\hat{\mathbf{x}} \cdot \frac{\hat{\mathbf{x}} - i\hat{\mathbf{y}}}{\sqrt{2}}\right|^2 + \left|(-\hat{\mathbf{y}}\cos\theta + \hat{\mathbf{z}}\sin\theta) \cdot \frac{\hat{\mathbf{x}} - i\hat{\mathbf{y}}}{\sqrt{2}}\right|^2 \\
&= 1 + \cos^2\theta + \sin^2\theta = 2
\end{aligned}
$$

Putting this into Eqn. (8) now gives:

$$
\begin{aligned}
w_{fi} &= \frac{1}{\hbar^3 c^3} \frac{\omega e^6}{2\pi} \frac{2^{10}}{3^9} \int_\Omega d\Omega \\
&= \frac{\omega e^6}{\hbar^3 c^3} \frac{2^{11}}{3^9} \\
&= \frac{\hbar\omega\alpha^4 c}{e^2} \frac{2^{11}}{3^9} \\
&= \frac{3e^2}{8a_0} \frac{\alpha^4 c}{e^2} \frac{2^{11}}{3^9} = \frac{2^8}{3^8} \frac{\alpha^4 c}{a_0} \\
&= \frac{2^8}{3^8} \left(\frac{1}{137}\right)^4 \frac{3 \times 10^8 \text{ m/s}}{0.53 \times 10^{-10} \text{ m}} = \frac{1}{1.6 \times 10^{-9} \text{ s}} \\
\tau &= 1.6 \times 10^{-9} \text{ s} \qquad (5.40.9)
\end{aligned}
$$

Chapter 6

6.1)

a) For N two state fermions in a harmonic oscillator we have

$$E = \sum_{n=0}^{n_{max}} 2E_n \tag{6.1.1}$$

Let us consider N to be odd. The highest occupied energy level will be $n = (N+1)/2 - 1$, populated with one fermion, but the highest filled level will be $n = (N-1)/2 - 1$. We then have for the energy

$$\begin{aligned}
E &= E_{(N-1)/2} + \sum_{n=0}^{(N-3)/2} 2E_n \\
&= \hbar\omega \left[\frac{N-1}{2} + \frac{1}{2} + 2 \sum_{n=0}^{(N-3)/2} \left(n + \frac{1}{2} \right) \right] \\
&= \hbar\omega \left[\frac{N-1}{2} + \frac{1}{2} + \frac{N-3}{2} + 1 + 2 \sum_{n=0}^{(N-3)/2} n \right] \\
&= \hbar\omega \left[\frac{N+1}{2} + \frac{1}{2} + \frac{N-3}{2} + 1 + \frac{(N-3)(N-1)}{4} \right] \\
&= \hbar\omega \frac{N^2+1}{4}
\end{aligned} \tag{6.1.2}$$

The fermi energy for odd N is the energy of the highest occupied level:

$$E_F = \hbar\omega \left(\frac{N-1}{2} + \frac{1}{2} \right) = \hbar\omega \frac{N}{2} \tag{6.1.3}$$

For even N we can just sum to the highest filled energy state because that will also be the highest occupied state.

$$\begin{aligned}
E &= 2\hbar\omega \sum_{n=0}^{N/2-1} \left(n + \frac{1}{2} \right) \\
&= \hbar\omega \left(\frac{N}{2} - 1 + 1 + \frac{N(N-2)}{4} \right) = \hbar\omega \frac{N^2}{4}
\end{aligned} \tag{6.1.4}$$

The Fermi energy for even N is

$$E_F = \hbar\omega \left(\frac{N}{2} - 1 + \frac{1}{2} \right) = \frac{\hbar\omega}{2}(N-1) \tag{6.1.5}$$

b) For $N \gg 1$, we can ignore the constant terms for both even and odd N cases, giving us

$$E = \hbar\omega \frac{N^2}{4} \tag{6.1.6}$$

$$E_F = \hbar\omega \frac{N}{2} \tag{6.1.7}$$

6.2)

Without angular orbital momenta, the spacial symmetry is even under particle exchange, therefore the spin states must also be symmetric. The possible J values are $\{2, 1, 0\}$ for two spin-1 particles, however, only the $J = \{2, 0\}$ states are symmetric. We must then restrict our states to exclude $J = 1$ in order to satisfy the Pauli-exclusion principle.

6.3)

For the s-orbitals of each energy level in the Helium atom, the spacial wave function is symmetric under particle exchange. If the electron were a *spinless* boson we would only be allowed symmetric spin states, and therefore require only symmetric space states (we cannot take anti-symmetric spin with anti-symmetric space states because there are no anti-symmetric spin states for a spinless particle). Following Eqns. (6.4.17)–(6.4.19) we conclude that we must choose $I + J$ for the the space function to remain symmetric. Sakurai explains that I and J are both positive quantities, therefore the sum is also a positive quantity. Taking E_i to be the hydrogenic energy of particle i and ΔE to be the interaction energy of the two particles, we have (cf. Eqn. (6.4.17))

$$E = E_1 + E_2 + \Delta E = E_1 + E_2 + I + J > E_1 + E_2 \tag{6.3.1}$$

thus the energy levels are shifted up.

6.5)

a) Note that for symmetric space function, we require a symmetric spin function for an overall symmetric state.

i) all three particles in an "up" state is easily given by

$$|\,\psi\,\rangle = |\,+++\,\rangle = |\,3,3\,\rangle \tag{6.5.1}$$

$J = 3$ is easily determined to be the J value for this state.

ii) by applying the lowering operator $J_- = J_{1-} + J_{2-} + J_{3-}$ and using the conventional coefficients $J_-|\,j,m\,\rangle = \sqrt{(j+m)(j-m+1)}|\,j,m-1\,\rangle$ we get a state consisting only of two particles in $|\,+\,\rangle$ and one particle in $|\,0\,\rangle$

$$|\,3,2\,\rangle = \frac{1}{\sqrt{3}}\left(|\,++0\,\rangle + |\,+0+\,\rangle + |\,0++\,\rangle\right) \tag{6.5.2}$$

For this state, $J = 3$ since the J is unchanged by the lowering operator.

iii) For future reference, let us construct the Young-Tableaux for this system, using $+ = 1$, $0 = 2$, and $- = 3$.

$$\underbrace{\begin{matrix} \boxed{1|1|1} & \boxed{1|1|2} & \boxed{1|1|3} & \boxed{1|2|2} & \boxed{1|2|3} \\ \boxed{1|3|3} & \boxed{2|2|2} & \boxed{2|2|3} & \boxed{2|3|3} & \boxed{3|3|3} \end{matrix}}_{\text{10 symmetric states}}$$

$$\underbrace{\begin{ytableau} 1 & 1 \\ 2 \end{ytableau}\quad\begin{ytableau} 1 & 2 \\ 2 \end{ytableau}\quad\begin{ytableau} 1 & 3 \\ 2 \end{ytableau}\quad\begin{ytableau} 1 & 1 \\ 3 \end{ytableau}\quad\begin{ytableau} 1 & 2 \\ 3 \end{ytableau}\quad\begin{ytableau} 1 & 3 \\ 3 \end{ytableau}\quad\begin{ytableau} 2 & 2 \\ 3 \end{ytableau}\quad\begin{ytableau} 2 & 3 \\ 3 \end{ytableau}}_{8\times 2 = 16 \text{ mixed states}}$$

$$\begin{ytableau} 1 \\ 2 \\ 3 \end{ytableau}$$

1 asymmetric state

We apply the lowering operator twice more to Eqn. (2) in order to obtain the $|\,3,0\rangle$ state.

$$\begin{aligned} |\,3,1\rangle &\propto |\,++-\,\rangle + 2|\,+00\rangle + 2|\,0+0\rangle + |\,+-+\,\rangle \\ &\qquad +2|\,00+\rangle + |\,-++\,\rangle \qquad (6.5.3) \\ |\,3,0\rangle &= \frac{1}{\sqrt{10}}\left(|\,+0-\rangle + |\,+-0\rangle + |\,-+0\rangle + |\,-0+\rangle\right. \\ &\qquad \left.+|\,0-+\rangle + |\,0+-\rangle + 2|\,000\rangle\,\right) \qquad (6.5.4) \end{aligned}$$

We wish to find a state given by

$$\begin{aligned} |\,\psi\rangle &= \frac{1}{\sqrt{6}}\left(|\,+0-\rangle + |\,+-0\rangle + |\,-+0\rangle + |\,-0+\rangle + |\,0-+\rangle + |\,0+-\rangle\right) \\ &= \sqrt{\frac{5}{3}}|\,3,0\rangle - \sqrt{\frac{2}{3}}|\,000\rangle \qquad (6.5.5) \end{aligned}$$

which is completely symmetric and has $m=0$. By Young-Tableaux, we know that there are 10 symmetric states which means that they consist only of $J=1$ and $J=3$ $(1\cdot 1+1+2\cdot 3+1=10)$. Therefore, This state must be a linear combination of $|\,3,0\rangle$ and $|\,1,0\rangle$ revealing its J values as $\{3,1\}$.

b) If the space part is anti-symmetric, we must have an anti-symmetric spin combination. This rules out any of the symmetric states, namely $|\,3,m\rangle$ and $|\,1,m\rangle$. We cannot construct an anti-symmetric state with all three particles in $|\,+\rangle$. Nor can we construct an anti-symmetric state with two particles $|\,+\rangle$ and one particle $|\,0\rangle$. By Young-Tableaux, There is only one completely anti-symmetric state, which must be the $|\,0,0\rangle$ state $(1\cdot 0+1=1)$. Also by Young-Tableaux, we know that this state consists only of all three particles in different states. $|\,0,0\rangle$ must be orthogonal to Eqn. (4) and be completely anti-symmetric. We must therefore thow out the $|\,000\rangle$ state which is entirely symmetric. In order to build this state we take all of the odd permutations of the top equation of Eqn. (5) and make them negative.

$$|\,0,0\rangle = \frac{1}{\sqrt{6}}\left(|\,+0-\rangle - |\,+-0\rangle + |\,-+0\rangle - |\,-0+\rangle + |\,0-+\rangle - |\,0+-\rangle\right) \qquad (6.5.6)$$

which is indeed anti-symmetric with all particles in different states, and therefore must be $|\,0,0\rangle$.

6.7)

a) A triplet state is symmetric, therefore we must have an anti-symmetric space state

$$\begin{aligned}\Psi_{nm} &= \frac{1}{\sqrt{2}}\left(\psi_n(x_1)\psi_m(x_2) - \psi_n(x_2)\psi_m(x_1)\right) &\quad(6.7.1)\\ &= \frac{\sqrt{2}}{L}\left(\sin\frac{n\pi}{L}x_1\ \sin\frac{m\pi}{L}x_2 - \sin\frac{n\pi}{L}x_2\ \sin\frac{m\pi}{L}x_1\right) &\quad(6.7.2)\end{aligned}$$

The ground state cannot have $n = m$ since this would yield $\Psi = 0$ (trivial solution). The lowest energy eigenstate is therefore $n = 1,\ m = 2$. The state and energy is

$$\begin{aligned}\Psi_{12} &= \frac{\sqrt{2}}{L}\left(\sin\frac{\pi}{L}x_1\ \sin\frac{2\pi}{L}x_2 - \sin\frac{\pi}{L}x_2\ \sin\frac{2\pi}{L}x_1\right) &\quad(6.7.3)\\ E_{12} &= \frac{\hbar^2\pi^2}{2mL^2}\left(1^2+2^2\right) = \frac{5\hbar^2\pi^2}{2mL^2} &\quad(6.7.4)\end{aligned}$$

b) The singlet state is anti-symmetric, so we must have a symmetric space state. We return to Eqn. (1) but use a $+$ instead of a $-$ in order to symmetrize the state. The ground state for the symmetric case, however, does not require that $n \neq m$. We then have

$$\begin{aligned}\Psi_{11} &= \frac{2}{L}\sin\frac{\pi}{L}x_1\ \sin\frac{\pi}{L}x_2 &\quad(6.7.5)\\ E_{11} &= \frac{\hbar^2\pi^2}{2mL^2}\left(1^2+1^2\right) = \frac{\hbar^2\pi^2}{mL^2} &\quad(6.7.6)\end{aligned}$$

c) To first order, the anti-symmetric state will give zero. consider

$$\begin{aligned}E_{12}^{(1)} &= \langle V\rangle = -\lambda\frac{2}{L^2}\int\int|\Psi_{12}(x_1,x_2)|^2\delta(x_1-x_2)\ dx_1\ dx_2\\ &= -\lambda\frac{2}{L^2}\int\int|\Psi_{12}(x_2,x_2)|^2\ dx_2 = 0 &\quad(6.7.7)\end{aligned}$$

since $\Psi_{12}(x_2,x_2)$ is trivially zero. The symmetric case gives

$$E_{11}^{(1)} = -\lambda\frac{4}{L}\int_0^L\sin^4\frac{\pi}{L}x_2\ dx_2 = -\frac{3}{2}\lambda \quad(6.7.8)$$

so that the energy of the symmetric space state with the spin singlet is lower than the anti-symmetric space state with the spin triplet.

Chapter 7

7.1)

a) For the one-dimensional case, the Lippmann-Schwinger approach should still be valid. For one dimension we take the incoming wave to be

$$\Phi(x) = \langle x \,|\, \phi \,\rangle = \frac{e^{ikx}}{\sqrt{2\pi}} \tag{7.1.1}$$

The scattered wave function is now

$$\Psi(x) \;=\; \langle x \,|\, \psi \,\rangle = \langle x \,|\, \phi \,\rangle + \int dx' \;\langle x \,|\frac{1}{E - H_0 + i\epsilon}|\, x' \,\rangle \,\langle x' \,|V|\, \psi \,\rangle \tag{7.1.2}$$

Let us first examine $\langle x \,|\frac{1}{E-H_0+i\epsilon}|\, x' \,\rangle$. Noting that in one dimension we have

$$\langle x \,|\, p \,\rangle \;=\; \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}} \tag{7.1.3}$$

$$H_0|\,\mathbf{p}\rangle \;=\; \frac{p^2}{2m}|\,\mathbf{p}\rangle \tag{7.1.4}$$

we can follow Sakurai's procedure, Eqns. (7.1.11)–(7.1.14).

$$\begin{aligned} G(x,x') \;&\equiv\; \frac{\hbar^2}{2m}\langle x \,|\frac{1}{E - H_0 + i\epsilon}|\, x' \,\rangle \qquad (7.1.5)\\ &=\; \frac{\hbar^2}{2m}\int_{-\infty}^{\infty} dp' \int_{-\infty}^{\infty} dp'' \;\frac{\delta(p'-p'')}{E - \frac{p^2}{2m} + i\epsilon}\,\frac{e^{ip'x/\hbar}}{\sqrt{2\pi\hbar}}\,\frac{e^{-ip''x'/\hbar}}{\sqrt{2\pi\hbar}} \\ &=\; \frac{\hbar^2}{2m}\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp' \;\frac{e^{ip'(x-x')/\hbar}}{E - \frac{p^2}{2m} + i\epsilon} \end{aligned}$$

and substituting $p' = \hbar q$ and $E = \hbar^2 k^2/2m$

$$\begin{aligned} G(x,x') \;&=\; \frac{\hbar^2}{2m}\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \hbar dq \;\frac{e^{iq(x-x')}}{\frac{\hbar^2 k^2}{2m} - \frac{\hbar^2 q^2}{2m} + i\epsilon} \\ &=\; -\frac{1}{2\pi}\int_{-\infty}^{\infty} dq \;\frac{e^{iq(x-x')}}{q^2 - k^2 + i\epsilon'} \end{aligned}$$

where $\epsilon' = 2m\epsilon/\hbar^2$. We see that there are two poles in the complex plane at $q = \pm\sqrt{k^2 + i\epsilon'} \approx \pm(k + i\epsilon'')$, if we let $\epsilon'' = \epsilon'/2k \ll 1$ (k is constant). This is valid since we will later be taking the limit as $\epsilon'' \to 0$. We can now perform the integral by closing it in the complex plane; closing in the upper-half plane for $x - x' > 0$ and in the lower-half plane for $x - x' < 0$. For the upper-half plane, only the $q = +k + i\epsilon''$ will be enclosed in the contour. We then have

$$\begin{aligned} G(x,x') \;&=\; -\lim_{\epsilon''\to 0}\frac{1}{2\pi}(2\pi i)\frac{e^{iq(x-x')}}{q + k + i\epsilon''}\bigg|_{q=k+i\epsilon''} \\ &=\; -\frac{i}{2}\frac{e^{ik(x-x')}}{k} \qquad (7.1.6) \end{aligned}$$

For the lower-half plane only $q = -k - i\epsilon''$ is enclosed and we get

$$\begin{aligned} G(x,x') &= -\lim_{\epsilon''\to 0} \frac{1}{2\pi}(-2\pi i)\left.\frac{e^{iq(x-x')}}{q-k-i\epsilon''}\right|_{q=-k-i\epsilon''} \\ &= -\frac{i}{2}\frac{e^{-ik(x-x')}}{k} \end{aligned} \tag{7.1.7}$$

Thus we can write

$$G(x,x') = -\frac{i}{2}\frac{e^{ik|x-x'|}}{k} \qquad \forall\ x \tag{7.1.8}$$

as the appropriate form for the Green's function. The integral equation becomes

$$\Psi(x) = \frac{e^{ikx}}{\sqrt{2\pi}} - \frac{im}{\hbar^2 k}\int dx'\ e^{ik|x-x'|}\,\langle\, x'\,|V|\,\psi\,\rangle \tag{7.1.9}$$

Note that for $x - x' > 0$, we only have expressions with e^{+ikx} (transmitted wave), while for $x - x' < 0$ we have both e^{+ikx} (incoming) and e^{-ikx} (reflected) waves.

b) If we take $V(x) = -(\gamma\hbar^2/2m)\delta(x)$, we can evaluate the integral as follows.

$$\begin{aligned} \Psi(x) &= \frac{e^{ikx}}{\sqrt{2\pi}} - \frac{im}{\hbar^2 k}\int_{-\infty}^{\infty} dx'\ e^{ik|x-x'|}\,\langle\, x'\,|V|\,\psi\,\rangle \\ &= \frac{e^{ikx}}{\sqrt{2\pi}} + \frac{i\gamma}{2k}\int_{-\infty}^{\infty} dx'\ e^{ik|x-x'|}\delta(x')\Psi(x') \\ &= \frac{e^{ikx}}{\sqrt{2\pi}} + \frac{i\gamma}{2k}e^{ik|x|}\Psi(0) \end{aligned} \tag{7.1.10}$$

Setting $x = 0$ and solving for $\Psi(0)$ yields

$$\Psi(0) = \frac{1}{\sqrt{2\pi}}\frac{1}{1-\frac{i\gamma}{2k}} \tag{7.1.11}$$

Inserting into Eqn. (10) gives us $\Psi(x)$ from which we can get the transmission and reflection probabilities.

$$\Psi(x) = \begin{cases} \dfrac{e^{ikx}}{\sqrt{2\pi}} + \dfrac{i\gamma}{2k}\dfrac{e^{-ikx}}{\sqrt{2\pi}}\dfrac{1}{1-\frac{i\gamma}{2k}} & x < 0 \\[2ex] \dfrac{e^{ikx}}{\sqrt{2\pi}} + \dfrac{i\gamma}{2k}\dfrac{e^{ikx}}{\sqrt{2\pi}}\dfrac{1}{1-\frac{i\gamma}{2k}} & x > 0 \end{cases} \tag{7.1.12}$$

$$\begin{aligned} T &= \frac{|\Psi(x>0)|^2}{|\Phi(x)|^2} \\ &= \left|1 + \frac{i\gamma}{2k}\frac{1}{1-\frac{i\gamma}{2k}}\right|^2 \\ &= \frac{1}{1+\left(\frac{\gamma}{2k}\right)^2} \end{aligned} \tag{7.1.13}$$

$$
\begin{aligned}
R &= \frac{|\Psi(x<0)_{reflected}|^2}{|\Phi(x)|^2} \\
&= \left|\frac{i\gamma}{2k}\frac{1}{1-\frac{i\gamma}{2k}}\right|^2 \\
&= \frac{(\gamma/2k)^2}{1+\left(\frac{\gamma}{2k}\right)^2}
\end{aligned} \tag{7.1.14}
$$

c) We observe that both T and R have poles for $k = i\gamma/2$. This motivates us to write

$$
\begin{aligned}
\Psi_b(x) &= \left[1+\left(\frac{\gamma}{2k}\right)^2\right]\Psi(x)\bigg|_{k=i\gamma/2} \\
&= \left[1+\left(\frac{\gamma}{2k}\right)^2\right]\frac{e^{ikx}}{\sqrt{2\pi}}+\frac{i\gamma}{2k}\frac{e^{ik|x|}}{\sqrt{2\pi}}\bigg|_{k=i\gamma/2} \\
&= \frac{e^{-\gamma|x|/2}}{\sqrt{2\pi}}
\end{aligned} \tag{7.1.15}
$$

This is a bound state with energy

$$
E = \frac{\hbar^2 k^2}{2m} = -\frac{\hbar^2\gamma^2}{4m} \tag{7.1.16}
$$

7.3)

For a potential $|V_0| \ll E$, we have a high energy situation, therefore the Born approximation is valid. For $kR \ll 1$ we can expand the exponential term in the expression for the scattering amplitude and take just the first few terms as follows.

$$
f^{(1)}(\theta) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}V_0\int d^3x'\ e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}'} \tag{7.3.1}
$$

$$
\approx -\frac{1}{4\pi}\frac{2m}{\hbar^2}V_0\int d^3x'\left(1+i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}'-\frac{1}{2}\left((\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}'\right)^2+\cdots\right) \tag{7.3.2}
$$

Let us look at just the first term and then compute the total cross-section.

$$
\begin{aligned}
f^{(1)}(\theta) &= -\frac{1}{4\pi}\frac{2m}{\hbar^2}V_0 4\pi\int_0^R r'^2\,dr' \\
&= -\frac{2m}{\hbar^2}\frac{V_0R^3}{3}
\end{aligned} \tag{7.3.3}
$$

$$
\begin{aligned}
\sigma_{tot} &= \int\frac{d\sigma}{d\Omega}\,d\Omega = \int|f^{(1)}(\theta)|^2 d\Omega \\
&= \left(\frac{2m}{\hbar^2}\frac{V_0R^3}{3}\right)^2\int d\Omega \\
\sigma_{tot} &= \left(\frac{16\pi}{9}\right)\frac{m^2V_0^2R^6}{\hbar^4}
\end{aligned} \tag{7.3.4}
$$

If the energy is slightly increased such that kR is also larger, but still much less than one, we may want to keep a few more terms in the exponential expansion of Eqn. (2). We can then write, using $q^2 = |\mathbf{k} - \mathbf{k}'|^2 = 2k^2(1 - \cos\theta)$,

$$\begin{aligned} f^{(1)}(\theta) &= -\frac{1}{4\pi}\frac{2m}{\hbar^2}V_0 2\pi \int_{-1}^{1} d(\cos\theta') \int_0^R \left(1 + iqr'\cos\theta' - \frac{1}{2}q^2 r'^2 \cos^2\theta'\right) \\ &= -\frac{2m}{\hbar^2}V_0 \int_0^R r'^2 - \frac{q^2 r'^4}{3}\, dr' \\ &= -\frac{2mV_0}{\hbar^2}\frac{R^3}{3}\left(1 - \frac{q^2R^2}{5}\right) \\ &= -\frac{2mV_0}{\hbar^2}\frac{R^3}{3}\left(1 - \frac{2k^2R^2}{5}(1-\cos\theta)\right) \end{aligned}$$

We can then write the differential cross-section (keeping only orders of $(kR)^2$).

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= |f^{(1)}(\theta)|^2 \\ &\approx \left(\frac{2mV_0}{\hbar^2}\frac{R^3}{3}\right)^2 \left(1 - \frac{4k^2R^2}{5}(1-\cos\theta) + \cdots\right) \\ &= \left(\frac{2mV_0}{\hbar^2}\frac{R^3}{3}\right)^2 \left[\left(1 - \frac{4k^2R^2}{5}\right) + \frac{4k^2R^2}{5}\cos\theta\right] \end{aligned} \tag{7.3.5}$$

Thus $d\sigma/d\Omega = A + B\cos\theta$, where

$$A = \left(\frac{2mV_0}{\hbar^2}\frac{R^3}{3}\right)^2 \left(1 - \frac{4k^2R^2}{5}\right) \tag{7.3.6}$$

$$B = \left(\frac{2mV_0}{\hbar^2}\frac{R^3}{3}\right)^2 \frac{4k^2R^2}{5} \tag{7.3.7}$$

7.4)

a) From Eqn. (7.6.16) from the text, and $|\delta_l| \ll 1$, we can say

$$f_l = \frac{e^{i\delta_l}\sin\delta_l}{k} \approx \frac{\delta_l}{k} \tag{7.4.1}$$

Then using Eqn. (7.6.17), the partial wave expantion, Eqn. (7.2.6), and $q^2 = 2k^2(1 - \cos\theta)$, we have

$$f(\theta) = \sum_l (2l+1)\frac{\delta_l}{k}P_l(\cos\theta) = -\frac{mV_0}{\hbar^2\mu k^2}\frac{1}{1 + \mu^2/2k^2 - \cos\theta}$$

Solving for δ_l by using orthogonality properties of spherical Bessel functions, we then have

$$\int_{-1}^{1}\sum_l (2l+1)\frac{\delta_l}{k}P_{l'}(\cos\theta)P_l(\cos\theta)\, d(\cos\theta) = -\frac{mV_0}{\hbar^2\mu k^2}\int_{-1}^{1}\frac{P_{l'}(\cos\theta)}{1+\mu^2/2k^2 - \cos\theta}\, d(\cos\theta)$$

$$
\begin{aligned}
\frac{2\delta_l}{k} &= -\frac{mV_0}{\hbar^2\mu k^2}\int_{-1}^{1}\frac{P_l(\cos\theta)}{1+\mu^2/2k^2-\cos\theta}\,d(\cos\theta) \\
\delta_l &= -\frac{mV_0}{\hbar^2\mu k}\left[\frac{1}{2}\int_{-1}^{1}\frac{P_l(\cos\theta)}{1+\mu^2/2k^2-\cos\theta}\,d(\cos\theta)\right] \\
&= -\frac{mV_0}{\hbar^2\mu k}\left[\frac{1}{2}\int_{-1}^{1}\frac{P_l(\zeta')}{\zeta-\zeta'}\,d\zeta'\right] \\
\delta_l &= -\frac{mV_0}{\hbar^2\mu k}Q_l(\zeta) \qquad (7.4.2)
\end{aligned}
$$

where $\zeta' = \cos\theta$ and $\zeta = 1 + \mu^2/2k^2$.

b) **i)** We see from the expression given in the text for $Q_l(\zeta)$ that $Q_l(\zeta) > 0$ for $\zeta > 0$, which it is. Thus Eqn. (2) shows that $\delta_l < 0$ for $V_0 > 0$ and $\delta_l > 0$ for $V_0 < 0$

ii) The de Broglie wavelength is $\lambda = 2\pi/k$, and the range of the Yukawa potential is $1/\mu$; for $\lambda \ll 1/\mu$ we can say $1/k \ll 1/\mu$ or $\mu/k \gg 1$. Keeping the first term in the expansion given in the text for $Q_l(\zeta)$, we can write

$$
\begin{aligned}
\delta_l &\approx -\frac{mV_0}{\hbar^2\mu k}\frac{l!}{1\cdot3\cdot5\cdots(2l+1)}\left(\frac{2k^2}{\mu^2}\right)^{l+1} \\
&\approx -\frac{mV_0}{\hbar^2\mu}\left(\frac{2}{\mu^2}\right)^{l+1}\frac{l!}{1\cdot3\cdot5\cdots(2l+1)}\,k^{2l+1} \qquad (7.4.3)
\end{aligned}
$$

Thus $\delta_l \propto k^{2l+1}$ with a proportionality constant

$$
-\frac{mV_0}{\hbar^2\mu}\left(\frac{2}{\mu^2}\right)^{l+1}\frac{l!}{1\cdot3\cdot5\cdots(2l+1)}
$$

7.6)

a) Using Eqn. (7.6.33) from the text, and the fact that $\Psi(a) = 0$, we can write

$$
\begin{aligned}
j_l(ka)\cos\delta_l &= n_l(ka)\sin\delta_l \\
\tan\delta_l &= \frac{j_l(ka)}{n_l(ka)}
\end{aligned}
$$

For S-wave scattering, $l = 0$, $j_0(ka) = \sin(ka)/ka$, and $n_0(ka) = -\cos(ka)/ka$ giving

$$
\begin{aligned}
\tan\delta_l &= -\tan(ka) \\
\delta_l &= -ka \qquad (7.6.1)
\end{aligned}
$$

b) We can use the result of part (a) and Eqn. (7.6.17) to calculate the total cross-section for the S-wave.

$$
f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l\,P_l(\cos\theta)
$$

$$
\begin{aligned}
f(\theta) &= -\frac{1}{k}e^{-ika}\sin ka \\
\sigma &= \int |f(\theta)|^2 d\Omega \\
&= \frac{\sin^2 ka}{k^2}4\pi
\end{aligned}
$$

Taking the limit as $k \to 0$

$$\sigma = 4\pi\frac{(ka)^2}{k^2} = 4\pi a^2 \tag{7.6.2}$$

This is 4 times the classical cross-section of πa^2.

Wallace09)

a) The scattering amplitude is given by

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1)f_l P_l(\cos\theta) \tag{Wallace09.1}$$

where $f_l = (S_l - 1)/2ik$. For a completely absorbing scatterer $r < R$ and no potential $r > R$, $S_l = 0(1)$ for $r < R(r > R)$. We therefore have

$$f(\theta) = \frac{i}{2k}\sum_{l=0}^{kR}(2l+1)P_l(\cos\theta) \tag{Wallace09.2}$$

We truncate the sum over l at kR because $f_l = 0$ for $l > kR$.

b) We can calculate the elastic scattering cross-section, σ_{el}, as follows.

$$
\begin{aligned}
\sigma_{el} &= \int d\Omega\, |f(\theta)|^2 && \text{(Wallace09.3)} \\
&= \int d\Omega\, \left|\frac{i}{2k}\sum_{l=0}^{kR}(2l+1)P_l(\cos\theta)\right|^2 \\
&= \frac{\pi}{k^2}\sum_{l=0}^{kR}(2l+1) \\
&= \frac{\pi}{k^2}\left(k^2R^2 + 2kR + 1\right) \\
\sigma_{el} &\approx \pi R^2 && \text{(Wallace09.4)}
\end{aligned}
$$

where we have used the orthogonality of P_l and $kR \gg 1$.

c) The total cross-section can be calculated as follows.

$$\sigma_t = \frac{4\pi}{k}\mathrm{Im}\{f(0)\} \tag{Wallace09.5}$$

$$\begin{aligned}
&= \frac{4\pi}{k}\mathrm{Im}\left\{ \frac{i}{2k}\sum_{l=0}^{kR}(2l+1)P_l(\cos\theta)\bigg|_{\theta=0} \right\} \\
&= \frac{2\pi}{k^2}\sum_{l=0}^{kR}(2l+1) \\
&= \frac{2\pi}{k^2}\left(k^2R^2 + 2kR + 1\right) \\
\sigma_t &\approx 2\pi R^2 \qquad \text{(Wallace09.6)}
\end{aligned}$$

Again, we have used $kR \gg 1$.

7.7)

a) From Eqn. (7.4.14) in the text, we find δ_l to be for $V = V_0 e^{-r^2/a_0^2}$

$$\begin{aligned}
\delta_l &= \Delta(b)|_{b=l/k} = -\frac{m}{2k\hbar^2}\int_{-\infty}^{\infty} V\left(\sqrt{l^2/k^2 + z'^2}\right)\,dz' \\
&= -\frac{m}{2k\hbar^2}\int_{-\infty}^{\infty} V_0 e^{-(l^2/k^2+z'^2)/a_0^2}\,dz' \\
&= -\frac{mV_0 e^{-l^2/k^2a_0^2}}{2k\hbar^2}\int_{-\infty}^{\infty} e^{-z'^2/a_0^2}\,dz' \\
\delta_l &= -\frac{mV_0 a_0\sqrt{\pi}}{2k\hbar^2}e^{-l^2/k^2a_0^2} \qquad (7.7.1)
\end{aligned}$$

We see from Eqn. (1) that $\delta_l \to 0$ very rapidly as $l \to \infty$

b) For the Yukawa potential $V = V_0 e^{-\mu r}/\mu r$, we get

$$\begin{aligned}
\delta_l &= \Delta(b)|_{b=l/k} = -\frac{m}{2k\hbar^2}\int_{-\infty}^{\infty} V\left(\sqrt{l^2/k^2 + z'^2}\right)\,dz' \\
&= -\frac{m}{2k\hbar^2}\int_{-\infty}^{\infty} V_0 \frac{e^{-\mu\sqrt{l^2/k^2+z'^2}}}{\mu\sqrt{l^2/k^2+z'^2}}\,dz'
\end{aligned}$$

This integral can be found in Gradshteyn & Ryzhik, *Table of Integrals Series and Products*, 4th Ed., p. 959. The result is $2K_0(\mu l/k)$.

$$\delta_l = -\frac{mV_0}{k\hbar^2\mu}K_0(\mu l/k) \qquad (7.7.2)$$

K_0 is the zeroth order hyperbolic Bessel function. As l gets large, $K_0 \to 0$ very rapidly, thus $\delta_l \to 0$ very rapidly.

7.9)

a) Using separation of variables, we know that $\Psi = \frac{u(r)}{r}Y_l^m$ satisfies Schrödinger's equation. We then have for $u(r)$, $l = 0$, and $V = \gamma\delta(r - R)$

$$\left[\nabla^2 + k^2\right]\frac{u(r)}{r} = \gamma\delta(r - R) \qquad (7.9.1)$$

Carrying out the Laplacian in r (spherical coordinates), and rearranging terms we get

$$\frac{d^2u}{dr^2} + \left[k^2 - \gamma\delta(r-R)\right] u = 0 \tag{7.9.2}$$

The general solution is then (considering $r \neq R$ and behavior at $r=0$ and $r=\infty$)

$$\Psi = \begin{cases} \dfrac{A}{r}\sin kr & r < R \\[2ex] \dfrac{C}{r}\sin kr - \dfrac{D}{r}\cos kr & r > R \end{cases} \tag{7.9.3}$$

Matching Ψ and the derivatives of $u(r)$ at the boundary $r = R$ gives

$$\begin{aligned} A\sin kR &= C\sin kR - D\cos kR & (7.9.4)\\ \frac{\gamma}{k}A\sin kR &= -A\cos kR + C\cos kR + D\sin kR & (7.9.5)\\ D &= -C\tan\delta_0 & (7.9.6) \end{aligned}$$

where Eqn. (6) is taken from the class lecture. Using Eqn. (6) we can write Eqns. (4) and (5) as

$$\begin{aligned} A\sin kR &= C'\sin(kR-\delta_0) & (7.9.7)\\ A\left[\frac{\gamma}{k}\cos kR + \sin kR\right] &= C'\cos(kR-\delta_0) & (7.9.8) \end{aligned}$$

Then dividing Eqn. (8) by Eqn. (7) gives

$$\cot(kR+\delta_0) = \frac{\gamma}{k}\cot kR + 1 \tag{7.9.9}$$

b) **i)** For $\gamma/k \gg 1$ and $\tan kR \neq 0$, the right hand side of Eqn. (9) is very large, thus the argument must be zero, in other words $\delta_0 = -kR$, which is the case for hard-sphere scattering.

ii) Trigonometric manipulation of Eqn. (9) gives

$$\cot\delta_0 = -\cot kR - \frac{k}{\gamma}\csc^2 kR \tag{7.9.10}$$

If $\cot\delta_0 = 0$ for resonance and $\tan kR \approx 0$, then

$$\begin{aligned} \tan kR &= -\frac{\gamma}{k}\sin^2 kR \\ \cos kR\sin kR &= -\frac{k}{\gamma} \\ kR - n\pi &\approx -\frac{kR}{\gamma R} \\ kR &= n\pi\left(1 - \frac{1}{\gamma R}\right) & (7.9.11) \end{aligned}$$

iii) For solutions inside a spherical potential well with boundaries $V(R)=\infty$, we have solutions given by Eqn. (3) for $r<R$. However, we have the restriction on k such that $k=n\pi/R$, in order to satisfy the boundary condition. This gives energies of

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2mR^2} \tag{7.9.12}$$

Comparing to energies from part (ii)

$$E_n \approx \frac{\hbar^2}{2m}\frac{n^2\pi^2}{R^2}\left(1-\frac{1}{\gamma R}\right)^2 \tag{7.9.13}$$

which agrees with the infinite well energies in the limit that $\gamma R \to \infty$.

iv)

$$\begin{aligned}\frac{d(\cot\delta_0)}{dE} &= \frac{dk}{dE}\frac{d(\cot\delta_0)}{dk} \\ &= \frac{dk}{dE}\left(1+\frac{2k}{\gamma}\cot kR - \frac{1}{\gamma R}\right) R\csc^2 kR\end{aligned}$$

Recall that $kR \approx n\pi - k/\gamma$, so that $\tan kR \approx \sin kR \approx k/\gamma$, thus

$$\begin{aligned}\frac{d(\cot\delta_0)}{dE} &= \frac{dk}{dE}\left(1-2-\frac{1}{\gamma R}\right) R\frac{\gamma^2}{k^2} \\ &\approx -\frac{dk}{dE}\frac{\gamma^2}{k^2}R = \frac{m\gamma^2 R}{\hbar^2 k^3} \end{aligned} \tag{7.9.14}$$

The resonance width Γ becomes

$$\begin{aligned}\Gamma &= \frac{-2}{\frac{d(\cot\delta_0)}{dE}} = \frac{2\hbar^2 k^3}{m\gamma^2 R} \\ &\approx \frac{2\hbar^2 n^3 \pi^3}{m\gamma^2 R^4}\end{aligned} \tag{7.9.15}$$

7.10)

From Eqn. (7.11.10) in the text we have

$$\begin{aligned}|\,\psi^{(+)}\rangle &= |\,k_i\rangle - \frac{i}{\hbar}\int_{-\infty}^{t} dt'\ e^{iH_0t'/\hbar}V(r,t')e^{\eta t'}e^{-iH_0t'/\hbar}|\,k_i\rangle \\ \langle k_f\,|\,\psi^{(+)}\,\rangle &= c_f^{(1)} = -\frac{i}{2\hbar}\,\langle k_f\,|V(r)|\,k_i\,\rangle\int_{-\infty}^{t} dt'\ e^{iE_ft'/\hbar}\left(e^{i\omega t'}+e^{-i\omega t'}\right)e^{-iE_it'/\hbar}|\,k_i\rangle \\ &= -\frac{i}{2\hbar}V_{fi}\int_{-\infty}^{t} dt'\ e^{i(E_f-E_i\pm\hbar\omega-i\hbar\eta)t'/\hbar} \\ &= -\frac{V_{fi}}{2}\frac{e^{i(E_f-E_i\pm\hbar\omega-i\hbar\eta)t/\hbar}}{E_f-E_i\pm\hbar\omega-i\hbar\eta}\end{aligned}$$

$$\begin{aligned} w_{fi} &= \frac{d}{dt}|c_{fi}^{(1)}|^2 \\ &= \lim_{\eta\to 0}\frac{|V_{fi}|^2}{4}\frac{d}{dt}\frac{e^{2\eta)t}}{(E_f - E_i \pm \hbar\omega)^2 + \hbar^2\eta^2} \\ &= \frac{\pi|V_{fi}|^2}{2\hbar}\delta\left[E_f - (E_i \pm \hbar\omega)\right] \end{aligned} \tag{7.10.1}$$

Thus the energy of the scattered particle is increased or decreased by $\hbar\omega$. The differential cross section is taken from p. 428 of the text.

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{\int w_{fi}\rho_n(E_f)\, dE_f}{incident\ flux} \\ &= \frac{\frac{V}{(2\pi)^3}\frac{k_f m}{\hbar^2}\frac{\pi|V_{fi}|^2}{2\hbar}}{\frac{\hbar k_i}{mV}} \\ &= \frac{V^2}{(2\pi)^2}\frac{k_f m^2}{k_i\hbar^4}\frac{|V_{fi}|^2}{4} \end{aligned} \tag{7.10.2}$$

If we box normalize the matrix elements of V_{fi} then we get

$$\frac{d\sigma}{d\Omega} = \frac{|V_{fi}|^2}{4(2\pi)^2}\frac{k_f m^2}{k_i\hbar^4} \tag{7.10.3}$$

7.11)

From Eqns. (7.12.13) and (7.12.15) from the text, and taking $Z = 1$, $n = 0$, and $|k| = |k'|$, we have

$$\frac{d\sigma}{d\Omega} = \frac{4m^2}{\hbar^4}\frac{e^4}{q^4}\left|-1 + F_0(\mathbf{q})\right|^2 \tag{7.11.1}$$

$$\begin{aligned} F_0(\mathbf{q}) &= \langle 0\,|e^{i\mathbf{q}\cdot\mathbf{x}_1}|\,0\,\rangle \\ &= \int d^3x_1\; e^{i\mathbf{q}\cdot\mathbf{x}_1}\left|\frac{2}{a_0^{3/2}e^{-r/a_0}\frac{1}{\sqrt{4\pi}}}\right|^2 \\ &= \frac{2}{a_0^3}\int_0^\infty r^2 e^{-2r/a_0}\; dr\int_{-1}^{1} e^{iqr\cos\theta}d(\cos\theta) \\ &= \frac{2}{iqa_0^3}\int_0^\infty r\left[e^{-(iq+2/a_0)} - e^{-(iq+2/a_0)}\right]\; dr \\ &= \frac{2}{iqa_0^3}\left[\frac{1}{(2/a_0 - iq)^2} - \frac{1}{(2/a_0 + iq)^2}\right] \\ &= \frac{16}{a_0^4}\frac{1}{4/a_0^2 + q^2} = \frac{16}{a_0^2}\frac{1}{4 + a_0^2 q^2} \end{aligned} \tag{7.11.2}$$

Thus the differential cross section becomes

$$\frac{d\sigma}{d\Omega} = \frac{4m^2}{\hbar^4}\frac{e^4}{q^4}\left[1 - \frac{16}{a_0^2}\frac{1}{4 + a_0^2 q^2}\right]^2 \tag{7.11.3}$$

Extra Wallace Problems

1)

a) The Klein-Gordon equation is

$$\left[\frac{1}{c^2}\frac{d^2}{dt^2} - \nabla^2 + \left(\frac{mc^2}{\hbar}\right)^2\right]\Psi = 0 \tag{1.1}$$

If we assume that Ψ is of the form $\exp[iEt/\hbar]\exp[\mathbf{k}\cdot\mathbf{r}]$, then the radial part of the equation becomes

$$-\frac{E^2}{\hbar^2 c^2} + \mathbf{k}^2 + \frac{m^2c^4}{\hbar^2c^2}$$

For the case of a spherical potential well of depth V_0, we can apply a scaler shift to the mass: $mc^2 \to mc^2 - V_0$. The potential acts only on the interval $0 < r < R$. Solving for k gives

$$k = \begin{cases} \pm\sqrt{\left(\frac{E}{\hbar c}\right)^2 - \left(\frac{mc^2 - V_0}{\hbar c}\right)^2} & r < R \\ \\ \pm\sqrt{\left(\frac{E}{\hbar c}\right)^2 - \left(\frac{mc^2}{\hbar c}\right)^2} & r > R \end{cases} \tag{1.2}$$

If we require bound states, then we must require that $k_<$ (k for $r < R$) yield an oscillating function, while $k_>$ yield a decaying function. These conditions will be met if E is in the range $mc^2 - V_0 < E < mc^2$. We can then write k as

$$k = \begin{cases} \pm\frac{|E|}{\hbar c}\sqrt{1 - \left(\frac{mc^2 - V_0}{E}\right)^2} & r < R \\ \\ \frac{i|E|}{\hbar c}\sqrt{\left(\frac{mc^2}{E}\right)^2 - 1} & r > R \end{cases} \tag{1.3}$$

These give solutions for Ψ

$$\begin{aligned} \Psi(r) &= \begin{cases} Ae^{-k_>(r-R)} & r > R \\ \\ Be^{ik_<r} + Ce^{-ik_<r} & r < R \end{cases} \\ &= \begin{cases} Ae^{ik_>(r-R)} & r > R \\ \\ 2iB\sin k_<r & r < R \end{cases} \end{aligned} \tag{1.4}$$

where the time dependence is understood and $k_{<,>}$ is given in Eqn. (3). The last step is achieved by requiring $\Psi(0) = 0$. Matching the two solutions and their derivatives at the boundary leads to transcendental condition on bound states.

$$\tan k_<R = -\frac{k_<}{|k_>|}. \tag{1.5}$$

We require one, and only one, bound state which is the zero energy bound state. Observing that Ψ is a sine function in the region $r < R$, we must require $k_< R > \pi/2$, else the derivative Ψ for $r < R$ will be positive at the boundary which would make it impossible to match to a decaying solution for $r > R$. We must also require that $k_< R < 3\pi/2$ in order to limit our bound states to just one, since the periodicity of Eqn. (5) is π. Note that the tangent gives infinities at $\pi/2, 3\pi/2$, which can only be achieved for $k_> = 0 \rightarrow E = mc^2$. Using this substitution and solving for V_0 in the expression $\pi/2 < k_< R < 3\pi/2$ gives the range on V_0 for which a single, zero energy bound state is possible.

$$mc^2 - \sqrt{m^2c^4 - \left(\frac{\pi\hbar c}{2R}\right)^2} < V_0 < mc^2 - \sqrt{m^2c^4 - \left(\frac{3\pi\hbar c}{2R}\right)^2} \tag{1.6}$$

b) If we treat the potential as the time part of a 4-vector potential, we simply offset the energy rather then the mass term: $E \rightarrow E - V_0$ instead of $mc^2 \rightarrow mc^2 - V_0$. Upon this exchange, in order to produce the appropriate wave function, our requirement on E now becomes $mc^2 + V_0 < E < mc^2$. The same requirements on $k_< R$ are maintained and Eqn. (5) is also preserved. The new range on V_0 becomes

$$mc^2 - \sqrt{m^2c^4 + \left(\frac{3\pi\hbar c}{2R}\right)^2} < V_0 < mc^2 - \sqrt{m^2c^4 + \left(\frac{\pi\hbar c}{2R}\right)^2} \tag{1.7}$$

2)

Using the expressions for α_i and β

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{2.1}$$

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.2}$$

we can show that $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$ and $\{\beta, \alpha_i\} = 0$.

$$\begin{aligned} \{\alpha_i, \alpha_j\} &= \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \\ &= \begin{pmatrix} \sigma_i\sigma_j + \sigma_j\sigma_i & 0 \\ 0 & \sigma_i\sigma_j + \sigma_j\sigma_i \end{pmatrix} \\ \{\alpha_i, \alpha_j\} &= 2\delta_{ij}\mathbb{1} \end{aligned} \tag{2.3}$$

$$\begin{aligned} \{\beta, \alpha_i\} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \{\beta, \alpha_i\} &= 0 \end{aligned} \tag{2.4}$$

Here we have used Eqn. (3.2.34) from the text for $\{\sigma_i, \sigma_j\}$ and $\mathbb{1}$ to indicate a 4×4 unit matrix. It is also useful to construct $[\alpha_i, \alpha_j]$ using $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$.

$$\begin{aligned}
[\alpha_i, \alpha_j] &= \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \\
&= \begin{pmatrix} \sigma_i\sigma_j - \sigma_j\sigma_i & 0 \\ 0 & \sigma_i\sigma_j - \sigma_j\sigma_i \end{pmatrix} \\
\{\alpha_i, \alpha_j\} &= 2i\epsilon_{ijk}\sigma_k\mathbb{1} \qquad (2.5)
\end{aligned}$$

Using the above relations we can write

$$\begin{aligned}
(\alpha \cdot \mathbf{a})(\alpha \cdot \mathbf{b}) &= \sum_{ij} \alpha_i\alpha_j a_i b_j \\
&= \frac{1}{2}\sum_{ij} \left(\{\alpha_i, \alpha_j\} + [\alpha_i, \alpha_j]\right) a_i b_j \\
&= \sum_{ij} (\delta_{ij} + i\epsilon_{ijk}\sigma_k) a_i b_j \\
(\alpha \cdot \mathbf{a})(\alpha \cdot \mathbf{b}) &= \mathbf{a} \cdot \mathbf{b}\mathbb{1} + i\sigma \cdot (\mathbf{a} \times \mathbf{b})\mathbb{1} \qquad (2.6)
\end{aligned}$$

3)

With the following definitions for $i = \{1, 2, 3\}$

$$\begin{aligned}
\gamma^0 &= \beta \qquad (3.1) \\
\gamma^i &= \beta\alpha_i \qquad (3.2)
\end{aligned}$$

we can show that for $\{\mu, \nu\} = \{0, 1, 2, 3\}$, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and $\frac{i}{2}[\gamma^\mu, \gamma^\nu] = \sigma^{\mu\nu}$ where $\sigma^{0i} = i\alpha_i$ and

$$\sigma^{ij} = \epsilon_{ijk}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \qquad (3.3)$$

For $\mu = 0, \nu = 0$ we get

$$\{\gamma^0, \gamma^0\} = \{\beta, \beta\} = 2\beta^2 = 2\mathbb{1} = 2g^{00}\mathbb{1} \qquad (3.4)$$

For $\mu = 0, \nu = i$ (similar for $\mu = i, \nu = 0$) we get

$$\begin{aligned}
\{\gamma^0, \gamma^i\} &= \{\beta, \beta\alpha_i\} \\
&= \beta^2\alpha_i + \beta\alpha_i\beta \\
&= \alpha_i - \alpha_i = 2g^{0i}\mathbb{1} = 2g^{i0}\mathbb{1} = 0 \qquad (3.5)
\end{aligned}$$

For $\mu = i, \nu = j$ we get

$$\begin{aligned}
\{\gamma^i, \gamma^j\} &= \{\beta\alpha_i, \beta\alpha_j\} \\
&= \beta\alpha_i\beta\alpha_j + \beta\alpha_j\beta\alpha_i\beta \\
&= -\alpha i\alpha j - \alpha j\alpha_i \\
&= -2\delta_{ij}\mathbb{1} = 2g^{ij}\mathbb{1} \qquad (3.6)
\end{aligned}$$

Thus $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. Following a similar procedure for the anti-commutation relation we get the following.

$$\begin{aligned}
\frac{i}{2}[\gamma^0, \gamma^0] &= \frac{i}{2}(\beta^2 - \beta^2) = 0 && (3.7)\\
\frac{i}{2}[\gamma^0, \gamma^i] &= \frac{i}{2}[\beta, \beta\alpha_i] \\
&= \frac{i}{2}(\alpha_i + \alpha_i) = i\alpha_i = \sigma^{0i} && (3.8)\\
\frac{i}{2}[\gamma^i, \gamma^j] &= \frac{i}{2}[\beta\alpha_i, \beta\alpha_j] \\
&= \frac{i}{2}(\beta\alpha_i\beta\alpha_j - \beta\alpha_j\beta\alpha_i\beta) \\
&= \frac{i}{2}(-\alpha i\alpha j + \alpha j\alpha_{i)} = -\frac{i}{2}[\alpha_i, \alpha_j] \\
&= \epsilon_{ijk}\sigma_k\mathbb{1} && (3.9)
\end{aligned}$$

Thus $\frac{i}{2}[\gamma^\mu, \gamma^\nu] = \sigma^{\mu\nu}$. Furthermore, we have

$$\begin{aligned}
\gamma^\mu a_\mu \gamma^\nu b_\nu &= \gamma^\mu\gamma^\nu a_\mu b_\nu \\
&= \frac{1}{2}\left(\{\gamma^\mu, \gamma^\nu\} + [\gamma^\mu, \gamma^\nu]\right) a_\mu b_\nu \\
&= (g^{\mu\nu} - i\sigma^{\mu\nu})a_\mu b_\nu \\
\gamma^\mu a_\mu \gamma^\nu b_\nu &= \mathbf{a}\cdot\mathbf{b}\mathbb{1} - i\sigma^{\mu\nu}a_\mu b_\nu\mathbb{1} && (3.10)
\end{aligned}$$

4)

a) For the time independent Dirac equation we can write

$$\begin{aligned}
Eu(p) &= (c\boldsymbol{\alpha}\cdot\mathbf{p} + \boldsymbol{\beta}mc^2)u(p) \quad \text{where} && (4.1)\\
\boldsymbol{\alpha}_i &= \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} && (4.2)\\
\boldsymbol{\beta} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} && (4.3)
\end{aligned}$$

Taking

$$u(p) = N\begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

Eqn. (1) becomes a set of two coupled equations.

$$\begin{aligned}
E\begin{pmatrix} \phi \\ \chi \end{pmatrix} &= \begin{pmatrix} mc^2 & c\boldsymbol{\sigma}\cdot\mathbf{p} \\ c\boldsymbol{\sigma}\cdot\mathbf{p} & -mc^2 \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} && (4.4)\\
(E - mc^2)\phi &= c\boldsymbol{\sigma}\cdot\mathbf{p}\chi && (4.5)\\
(E + mc^2)\phi &= c\boldsymbol{\sigma}\cdot\mathbf{p}\phi \quad \text{or} && (4.6)
\end{aligned}$$

$$\begin{aligned}\phi &= \frac{c\boldsymbol{\sigma}\cdot\mathbf{p}}{E-mc^2}\chi && (4.7)\\ \chi &= \frac{c\boldsymbol{\sigma}\cdot\mathbf{p}}{E+mc^2}\phi && (4.8)\end{aligned}$$

Inserting Eqn. (6) into Eqn. (5) or Eqn. (7) into Eqn. (4) gives the eigen-values.

$$\begin{aligned}(E^2-m^2c^4) &= c^2(\boldsymbol{\sigma}\cdot\mathbf{p})(\boldsymbol{\sigma}\cdot\mathbf{p})\\ E &= \pm\sqrt{\mathbf{p}^2c^2+m^2c^4} && (4.9)\end{aligned}$$

We take the convention

$$\begin{aligned}\phi_\pm &= \begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix} \qquad E>0 && (4.10)\\ \chi_\pm &= \begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix} \qquad E<0 && (4.11)\end{aligned}$$

Noting that

$$\begin{aligned}c\boldsymbol{\sigma}\cdot\mathbf{p}\begin{pmatrix}1\\0\end{pmatrix} &= c\begin{pmatrix}p_z\\p_x+ip_y\end{pmatrix} && (4.12)\\ c\boldsymbol{\sigma}\cdot\mathbf{p}\begin{pmatrix}0\\1\end{pmatrix} &= c\begin{pmatrix}p_x-ip_y\\-p_z\end{pmatrix} && (4.13)\end{aligned}$$

we can write the eigen-vectors of the Dirac equation as

$$u(p) = N\times\begin{cases}\begin{pmatrix}1\\0\\\frac{cp_z}{E+mc^2}\\\frac{c(p_x+ip_y)}{E+mc^2}\end{pmatrix},\begin{pmatrix}0\\1\\\frac{c(p_x-ip_y)}{E+mc^2}\\\frac{-cp_z}{E+mc^2}\end{pmatrix} & E>0\\[2ex] \begin{pmatrix}\frac{cp_z}{E-mc^2}\\\frac{c(p_x+ip_y)}{E-mc^2}\\1\\0\end{pmatrix},\begin{pmatrix}\frac{c(p_x-ip_y)}{E-mc^2}\\\frac{-cp_z}{E-mc^2}\\0\\1\end{pmatrix} & E<0\end{cases} \qquad (4.14)$$

b) Normalizing Eqn. (14) gives

$$\begin{aligned}u^\dagger u &= N^2\begin{pmatrix}1 & 0 & \frac{cp_z}{E+mc^2} & \frac{c(p_x-ip_y)}{E+mc^2}\end{pmatrix}\begin{pmatrix}1\\0\\\frac{cp_z}{E+mc^2}\\\frac{c(p_x+ip_y)}{E+mc^2}\end{pmatrix}\equiv 1\\ &= N^2\left(1+\frac{\mathbf{p}^2c^2}{(E+mc^2)^2}\right)\\ N &= \frac{1}{\sqrt{1+\frac{\mathbf{p}^2c^2}{(E+mc^2)^2}}} && (4.15)\end{aligned}$$

Clearly all vectors will have the same normalization constant, since for $E < 0$, the denominator is $(-|E| - mc^2)^2 = (|E| + mc^2)^2$.

c)

$$\begin{pmatrix} 1 & 0 & \frac{cp_z}{E+mc^2} & \frac{c(p_x - ip_y)}{E+mc^2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x - ip_y)}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \end{pmatrix} = $$
$$= \frac{p_z(p_x - ip_y)c^2}{(E+mc^2)^2} - \frac{p_z(p_x - ip_y)c^2}{(E+mc^2)^2} = 0 \quad (4.16)$$

Similar for the two $E < 0$ eigen-vectors. Now

$$\begin{pmatrix} 1 & 0 & \frac{cp_z}{E+mc^2} & \frac{c(p_x - ip_y)}{E+mc^2} \end{pmatrix} \begin{pmatrix} \frac{cp_z}{-|E|-mc^2} \\ \frac{c(p_x + ip_y)}{-|E|-mc^2 1} \\ 1 \\ 0 \end{pmatrix} = $$
$$= \frac{cp_z}{-|E| - mc^2} + \frac{cp_z}{|E| + mc^2} = 0 \quad (4.17)$$

Similar for all permutations of $E < 0$ and $E > 0$ eigen-vectors. Thus all eigen-vectors ore orthogonal.