Parameterizing toroidal surfaces

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January 5, 2018

1 Introduction

In this note we compare the VMEC and Garabedian representation of toroidal surfaces. The transformation between the coefficients of each representation is derived. A demonstration is given that for a given physical surface shape, the coefficients in either representation are not unique.

2 VMEC representation

In the VMEC code, the cylindrical coordinates (R, Z) are parameterized as functions of a poloidal angle θ and toroidal angle ζ using

$$R(\theta,\zeta) = \sum_{m,n} R^c_{m,n} \cos(m\theta - n\zeta) + R^s_{m,n} \sin(m\theta - n\zeta), \tag{1}$$
$$Z(\theta,\zeta) = \sum_{m,n} Z^c_{m,n} \cos(m\theta - n\zeta) + Z^s_{m,n} \sin(m\theta - n\zeta),$$

where $R^c_{m,n}$, $R^s_{m,n}$, $Z^c_{m,n}$, and $Z^s_{m,n}$ are coefficients that determine the surface shape.

Notice that $R^c_{m,n}$ and $Z^c_{m,n}$ give identical contributions to $R^c_{-m,-n}$ and $Z^c_{-m,-n}$, and $R^s_{m,n}$ and $Z^s_{m,n}$ give (-1) times the contributions of $R^s_{-m,-n}$ and $Z^s_{-m,-n}$. Therefore it is no loss of generality to consider only non-negative m, and for the $m = 0$ modes it is no loss of generality to consider only non-negative n.

If the surface is stellarator-symmetric about $(\theta, \zeta) = (0, 0)$, then flipping the sign of θ and ζ will leave R unchanged but will flip the sign of Z. In this case $R^s_{m,n} = 0$ and $Z^c_{m,n} = 0$ for all m and n.

3 Garabedian's representation

The representation introduced by Paul Garabedian is

$$R(\theta,\zeta) + iZ(\theta,\zeta) = e^{i\theta} \sum_{m,n} \Delta_{m,n} e^{-im\theta + in\zeta}, \tag{2}$$

where the parameters $\Delta_{m,n}$ determine the surface shape. (Here we sum over all integer values of m and n, including negative values.) If the surface is stellarator-symmetric about $(\theta, \zeta) = (0, 0)$, then flipping the sign of θ and ζ will leave R unchanged but will flip the sign of Z. Thus:

$$R(\theta,\zeta) - iZ(\theta,\zeta) = e^{-i\theta} \sum_{m,n} \Delta_{m,n} e^{im\theta - in\zeta}. \tag{3}$$

The complex conjugate of this relation is identical to (2), except with $\Delta_{m,n}$ replaced by its complex conjugate. Equating Fourier components to (2), we find $\Delta_{m,n} = \Delta^*_{m,n}$, that is, $\Delta_{m,n}$ is real. Hence, stellarator symmetry (about $(\theta, \zeta) = (0, 0)$) is equivalent to $\Delta_{m,n}$ being real.

3.1 Converting from VMEC to Garabedian coefficients

To relate the VMEC and Garabedian representations, we plug (1) into (2):

$$\sum_{m,n} \Delta_{m,n} e^{i(1-m)\theta+in\zeta} = \sum_{m,n} \left[(R^c_{m,n} + iZ^c_{m,n}) \cos(m\theta - n\zeta) + (R^s_{m,n} + iZ^s_{m,n}) \sin(m\theta - n\zeta) \right]. \quad (4)$$

Writing the cosine and sine functions in terms of complex exponentials,

$$\begin{aligned} \sum_{m,n} \Delta_{m,n} e^{i(1-m)\theta+in\zeta} =& \frac{1}{2} \sum_{m,n} \Big\{ (R^c_{m,n} + iZ^c_{m,n}) \left[e^{im\theta-in\zeta} + e^{-im\theta+in\zeta} \right] \\ &+ (-iR^s_{m,n} + Z^s_{m,n}) \left[e^{im\theta-in\zeta} - e^{-im\theta+in\zeta} \right] \Big\}. \end{aligned} \quad (5)$$

For the terms $\propto \exp(im\theta - in\zeta)$ on the right-hand side, we are free to replace the dummy index m with $-m$, noting that the sum over all m is equivalent to a sum over all $-m$, and similarly we can replace $n \to -n$. The result is

$$\begin{aligned} \sum_{m,n} \Delta_{m,n} e^{i(1-m)\theta+in\zeta} =& \frac{1}{2} \sum_{m,n} e^{-im\theta+in\zeta} \left[R^c_{-m,-n} + iZ^c_{-m,-n} + R^c_{m,n} + iZ^c_{m,n} \right. \\ & \left. -iR^s_{-m,-n} + Z^s_{-m,-n} + iR^s_{m,n} - Z^s_{m,n} \right]. \end{aligned} \quad (6)$$

Now on the right hand side we can replace the dummy index m with $m-1$, noting that the sum over all m is identical to a sum over all $m-1$. The result is

$$\begin{aligned} \sum_{m,n} \Delta_{m,n} e^{i(1-m)\theta+in\zeta} =& \frac{1}{2} \sum_{m,n} e^{i(1-m)\theta+in\zeta} \left[R^c_{1-m,-n} + iZ^c_{1-m,-n} + R^c_{m-1,n} + iZ^c_{m-1,n} \right. \\ & \left. -iR^s_{1-m,-n} + Z^s_{1-m,-n} + iR^s_{m-1,n} - Z^s_{m-1,n} \right]. \end{aligned} \quad (7)$$

Each Fourier mode of the left hand side must equal the corresponding Fourier mode on the right hand side, so

$$\Delta_{m,n} = \frac{1}{2} \left[R^c_{1-m,-n} + iZ^c_{1-m,-n} + R^c_{m-1,n} + iZ^c_{m-1,n} - iR^s_{1-m,-n} + Z^s_{1-m,-n} + iR^s_{m-1,n} - Z^s_{m-1,n} \right]. \quad (8)$$

Specializing now to stellarator symmetry, the left side is real, and the quantities multiplied by i on the right side are each 0, leaving

$$\Delta_{m,n} = \frac{1}{2} \left[R^c_{1-m,-n} + R^c_{m-1,n} + Z^s_{1-m,-n} - Z^s_{m-1,n} \right]. \quad (9)$$

3.2 Converting from Garabedian to VMEC coefficients

Here we work out the conversion from the Garabedian to VMEC representation for the case of stellarator symmetry. First, we replace $n \to -n$ and $m \to 2-m$ in (9):

$$\Delta_{2-m,-n} = \frac{1}{2} \left[R^c_{m-1,n} + R^c_{1-m,-n} + Z^s_{m-1,n} - Z^s_{1-m,-n} \right]. \quad (10)$$

Adding (9) to (10),

$$R^c_{1-m,-n} + R^c_{m-1,n} = \Delta_{m,n} + \Delta_{2-m,-n}. \quad (11)$$

When $m = 1$ and $n = 0$, (11) gives

$$R^c_{0,0} = \Delta_{1,0}. \quad (12)$$

When $m=1$ and $n>0$, (11) gives

$$R^c_{0,n} = \Delta_{1,n} + \Delta_{1,-n}. \tag{13}$$

The same result follows if $m=1$ and $n<0$ in (11). If $m>1$, then the first term of (11) vanishes. We can then replace $m \to m+1$ in the remaining terms, giving

$$R^c_{m,n} = \Delta_{1-m,-n} + \Delta_{1+m,n}. \tag{14}$$

The same result follows if $m<1$ in (11). Equations (12)-(14) give the VMEC $R^c_{m,n}$ coefficients in terms of the Garabedian coefficients.

Next we derive similar relations for the $Z^s_{m,n}$ coefficients. Subtracting (10) from (9) gives

$$Z^s_{1-m,-n} - Z^s_{m-1,n} = \Delta_{m,n} - \Delta_{2-m,-n}. \tag{15}$$

If $m=1$ and $n=0$, this expression reduces to $0=0$. If $m=1$ and $n>0$, (15) gives

$$Z^s_{0,n} = \Delta_{1,-n} - \Delta_{1,n}. \tag{16}$$

This same result is obtained if $m=1$ and $n<0$ in (15). If $m>1$, the first term in (15) vanishes. Substituting $m \to m+1$ in the remaining terms, we find

$$Z^s_{m,n} = \Delta_{1-m,-n} - \Delta_{1+m,n}. \tag{17}$$

The same equation results if $m<1$ in (15). Equations (16)-(17) give the VMEC $Z^s_{m,n}$ coefficients in terms of the Garabedian coefficients.

3.3 Counting degrees of freedom

Given a finite number of nonzero VMEC coefficients $R^c_{m,n}$ and $Z^s_{m,n}$, (9) indicates that there is an exactly equivalent Garabedian representation with a finite number of nonzero $\Delta_{m,n}$ coefficients. Considering that the VMEC coefficients vanish for $m<0$, (9) indicates that the $\Delta_{m,n}$ coefficients will need to be nonzero for negative m. The Garabedian representation requires twice as many m values as the VMEC representation, but the Garabedian representation also requires half as many quantities for each m and n (a single $\Delta_{m,n}$, compared to the two quantities $R^c_{m,n}$ and $Z^s_{m,n}$ for the VMEC representation.) Hence, the number of degrees of freedom required to represent a given shape is (at least roughly) the same.

4 Non-uniqueness

4.1 VMEC representation

For a given surface shape, the VMEC coefficients $R^c_{m,n}$ and $Z^s_{m,n}$ (and $R^s_{m,n}$ and $Z^c_{m,n}$ if the shape is not stellarator-symmetric) are not unique. For the particular case of a circular cross-section in the poloidal plane, we now demonstrate that we can can specify an infinite family of different $\{R^c_{m,n}, Z^s_{m,n}\}$ values that all yield the same shape. For simplicity, let us neglect the toroidal direction and shift R by the major radius, so we can consider the circle $R^2+Z^2=1$. We replace $R^c_{m,n} \to R_m$ and $Z^s_{m,n} \to Z_m$ to simplify notation. One set of VMEC coefficients that corresponds to this shape is

$$\begin{aligned} R_m &= 1 \text{ if } m=1, \text{ otherwise } R_m = 0, \\ Z_m &= 1 \text{ if } m=1, \text{ otherwise } Z_m = 0. \end{aligned} \tag{18}$$

However, we can obtain other VMEC coefficients for the same surface if we parameterize the circle using a different poloidal angle ϑ related to the original angle θ by

$$\theta = \vartheta - \alpha \sin \vartheta, \tag{19}$$

where α is some constant. The circle can then be written as

$$\begin{aligned} R &= \cos\left(\vartheta - \alpha \sin \vartheta\right), \\ Z &= \sin\left(\vartheta - \alpha \sin \vartheta\right). \end{aligned} \tag{20}$$

We now write the VMEC representation of the shape in terms of the poloidal angle ϑ, adding a superscript $^{\alpha}$ to R_m and Z_m:

$$\begin{aligned} R &= \sum_m R_m^{\alpha} \cos(m\vartheta), \\ Z &= \sum_m Z_m^{\alpha} \sin(m\vartheta). \end{aligned} \tag{21}$$

Equating (20)-(21), and expressing the cosine and sine functions as complex exponentials,

$$\begin{aligned} \sum_m R_m^{\alpha} \left(e^{im\vartheta} + e^{-im\vartheta}\right) &= e^{i(\vartheta - \alpha \sin\vartheta)} + e^{-i(\vartheta - \alpha \sin\vartheta)}, \\ \sum_m Z_m^{\alpha} \left(e^{im\vartheta} - e^{-im\vartheta}\right) &= e^{i(\vartheta - \alpha \sin\vartheta)} - e^{-i(\vartheta - \alpha \sin\vartheta)}. \end{aligned} \tag{22}$$

We next apply the operation

$$\frac{1}{2\pi} \int_0^{2\pi} d\vartheta \; e^{-iM\vartheta} (\ldots), \tag{23}$$

where M is any integer. The right-hand sides can be evaluated using

$$J_n(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} d\vartheta \; e^{i(n\vartheta - \alpha \sin\vartheta)}, \tag{24}$$

where J_n is the Bessel function. Thus,

$$\begin{aligned} R_M^{\alpha} + R_{-M}^{\alpha} &= J_{1-M}(\alpha) + J_{-1-M}(-\alpha), \\ Z_M^{\alpha} - Z_{-M}^{\alpha} &= J_{1-M}(\alpha) - J_{-1-M}(-\alpha). \end{aligned} \tag{25}$$

When $M = 0$, (25) and the identity

$$J_n(-\alpha) = J_{-n}(\alpha) \tag{26}$$

imply

$$R_0^{\alpha} = J_1(\alpha). \tag{27}$$

(The term Z_0^{α} multiplies $\sin 0 = 0$ so is not used.) For $M > 1$, the second left-hand side term of each equation in (25) is defined to be 0, leaving

$$\begin{aligned} R_M^{\alpha} &= J_{1-M}(\alpha) + J_{1+M}(\alpha), \\ Z_M^{\alpha} &= J_{1-M}(\alpha) - J_{1+M}(\alpha). \end{aligned} \tag{28}$$

For any value of α, equations (27)-(28) provide a different set of $\{R_m, Z_m\}$ coefficients for the unit circle, demonstrating the coefficients are not unique.

The sequences R_m^{α} and Z_m^{α} grow exponentially small with m, with $|R_m^{\alpha}|$ and $|Z_m^{\alpha}|$ both smaller than 10^{-15} for $m > 15$ when $|\alpha| \leq 1$.

4.2 Garabedian representation

Similarly, the Garabedian coefficients $\Delta_{m,n}$ are not unique, as we can demonstrate with the same example. We apply (9) to the R_m^α and Z_m^α coefficients derived above. We again neglect the toroidal direction, so we write $\Delta_{m,n} \rightarrow \Delta_m^\alpha$. For $m = 1$,

$$\Delta_1^\alpha = R_0^\alpha = J_1(\alpha). \tag{29}$$

For $m > 1$,

$$\Delta_m^\alpha = \frac{1}{2}(R_{m-1}^\alpha - Z_{m-1}^\alpha) = J_{-m}(-\alpha) = J_m(\alpha), \tag{30}$$

where (26) has been applied. For $m < 1$,

$$\Delta_m^\alpha = \frac{1}{2}(R_{1-m}^\alpha + Z_{1-m}^\alpha) = J_m(\alpha). \tag{31}$$

Thus, we find $\Delta_m^\alpha = J_m(\alpha)$ for all m. This result demonstrates an infinite set of Garabedian coefficients that all describe the same shape, hence the Garabedian representation is not unique.