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Canonical angular momentum conservation in ideal MHD

In this document I discuss a conservation equation which applies only to plasmas whose $\mathbf{E}$ and $\mathbf{B}$ fields remain axisymmetric over time, although any other field (e.g. density, velocity, pressure, …) may be nonaxisymmetric. In this document I present two derivations of the conservation law. The first derivation shows what *fluid* conservation law we should expect as a consequence of conservation of canonical angular momentum in the *single-particle* picture; the second derivation shows how the same result can be found directly from the ideal MHD equations.

1. Derivation from single-particle canonical momentum conservation

Suppose the electrostatic potential and the cylindrical components of the vector potential are axisymmetric:

$$0=\frac{\partial \phi}{\partial \zeta}=\frac{\partial A_z}{\partial \zeta}=\frac{\partial A_\zeta}{\partial \zeta}=\frac{\partial A_R}{\partial \zeta} \tag{1}$$

We allow each of these potentials to be time-dependent. Then, if there are no collisions, the canonical angular momentum of each particle j of species α should be time-independent:

$$p_\zeta^{(j)}=m_\alpha R\left(\mathbf{r}^{(j)}(t)\right)v_\zeta^{(j)}(t)+\frac{q_\alpha}{c}R\left(\mathbf{r}^{(j)}(t)\right)A_\zeta\left(\mathbf{r}^{(j)}(t)\right)=\text{independent of time} \tag{2}$$

Now consider a small box of size $\delta x\times\delta y\times\delta z$ at a time-independent position $\mathbf{r}$. The change in the total canonical angular momentum of the species inside the box during a small interval of time Δt can be expressed as follows:

$$\Delta\left\{\sum_{j \text{ in box at } \mathbf{r}} p_\zeta^{(j)}\right\}=\sum_{\substack{j \text{ which leave}\\ \text{box during } \Delta t}} p_\zeta^{(j)} \tag{3}$$

The quantity in braces above can be written

$$\sum_{j \text{ in box at } \mathbf{r}} p_\zeta^{(j)}=\sum_{j \text{ in box at } \mathbf{r}}\left\{m_\alpha R(\mathbf{r})v_\zeta^{(j)}(t)+\frac{q_\alpha}{c}R(\mathbf{r})A_\zeta(\mathbf{r})\right\}=\left[m_\alpha R n_\alpha u_{\alpha\zeta}+\frac{q_\alpha}{c}R n_\alpha A_\zeta\right]\delta x\ \delta y\ \delta z \tag{4}$$

To evaluate the right-hand side of (3), we add up the particles whose x coordinates are within $v_x\Delta t$ of the box's constant-x walls, and similarly for y and z:

$$\begin{aligned}\sum_{\substack{j \text{ which leave}\\ \text{box during } \Delta t}} p_\zeta^{(j)} &= \sum_{\substack{j \text{ which leave through}\\ \text{constant } x+\delta x \text{ face}}} p_\zeta^{(j)} - \sum_{\substack{j \text{ which enter through}\\ \text{constant } x \text{ face}}} p_\zeta^{(j)} + (y,\ z\ \text{ faces})\\ &= \sum_{\substack{j \text{ which leave}\\ \text{constant } x+\delta x \text{ face}}} v_x^{(j)}\Delta t\left[m_\alpha R(\mathbf{r}+\delta x\ \mathbf{e}_x)v_\zeta^{(j)}(t)+\frac{q_\alpha}{c}R(\mathbf{r}+\delta x\ \mathbf{e}_x)A_\zeta(\mathbf{r}+\delta x\ \mathbf{e}_x)\right]\\ &\quad-\sum_{\substack{j \text{ which enter}\\ \text{constant } x \text{ face}}} v_x^{(j)}\Delta t\left[m_\alpha R(\mathbf{r})v_\zeta^{(j)}(t)+\frac{q_\alpha}{c}R(\mathbf{r})A_\zeta(\mathbf{r})\right]+(y,\ z\ \text{faces})\end{aligned} \tag{5}$$

…

Continuing,

$$\sum_{\substack{j \text{ which leave} \\ \text{box during } \Delta t}} p_\zeta^{(j)} = \int d^3v \sum_{\substack{j \text{ which leave} \\ \text{constant } x+\delta x \text{ face}}} v_x \Delta t \left[m_\alpha R(\mathbf{r}+\delta x\, \mathbf{e}_x) v_\zeta(t) + \frac{q_\alpha}{c} R(\mathbf{r}+\delta x\, \mathbf{e}_x) A_\zeta(\mathbf{r}+\delta x\, \mathbf{e}_x) \right] \delta\left(\mathbf{v}-\mathbf{v}^{(j)}\right) \tag{6}$$

$$-\int d^3v \sum_{\substack{j \text{ which enter} \\ \text{constant } x \text{ face}}} v_x \Delta t \left[m_\alpha R(\mathbf{r}) v_\zeta(t) + \frac{q_\alpha}{c} R(\mathbf{r}) A_\zeta(\mathbf{r}) \right] \delta\left(\mathbf{v}-\mathbf{v}^{(j)}\right) + (y,\ z \text{ faces})$$

$$= \delta y\ \delta z \int d^3v\, f(\mathbf{r}+\delta x\, \mathbf{e}_x, \mathbf{v}, t) v_x \Delta t \left[m_\alpha R(\mathbf{r}+\delta x\, \mathbf{e}_x) v_\zeta(t) + \frac{q_\alpha}{c} R(\mathbf{r}+\delta x\, \mathbf{e}_x) A_\zeta(\mathbf{r}+\delta x\, \mathbf{e}_x) \right]$$

$$-\delta y\ \delta z \int d^3v\, f(\mathbf{r}, \mathbf{v}, t) v_x \Delta t \left[m_\alpha R(\mathbf{r}) v_\zeta(t) + \frac{q_\alpha}{c} R(\mathbf{r}) A_\zeta(\mathbf{r}) \right] + (y,\ z \text{ faces})$$

$$= \Delta t\ \delta y\ \delta z \left\{ \begin{array}{l} \left[u_{\alpha x} m_\alpha R n_\alpha u_{\alpha\zeta} + u_{\alpha x} \frac{q_\alpha}{c} R n_\alpha A_\zeta \right]_{\mathbf{r}+\delta x\, \mathbf{e}_x} \\ -\left[u_{\alpha x} m_\alpha R n_\alpha u_{\alpha\zeta} + u_{\alpha x} \frac{q_\alpha}{c} R n_\alpha A_\zeta \right]_{\mathbf{r}} \end{array} \right\} + (y,\ z \text{ faces})$$

$$= \Delta t\ \delta x\ \delta y\ \delta z\ \frac{\partial}{\partial x} \left[m_\alpha R n_\alpha u_{\alpha\zeta} u_{\alpha x} + \frac{q_\alpha}{c} R n_\alpha A_\zeta u_{\alpha x} \right] + (y,\ z \text{ faces})$$

$$= \Delta t\ \delta x\ \delta y\ \delta z\ \nabla \cdot \left[m_\alpha R n_\alpha u_{\alpha\zeta} \mathbf{u}_\alpha + \frac{q_\alpha}{c} R n_\alpha A_\zeta \mathbf{u}_\alpha \right]$$

Therefore,

$$\delta x\ \delta y\ \delta z \left[m_\alpha R n_\alpha u_{\alpha\zeta} + \frac{q_\alpha}{c} R n_\alpha A_\zeta \right] = \Delta t\ \delta x\ \delta y\ \delta z\ \nabla \cdot \left[m_\alpha R n_\alpha u_{\alpha\zeta} \mathbf{u}_\alpha + \frac{q_\alpha}{c} R n_\alpha A_\zeta \mathbf{u}_\alpha \right] \tag{7}$$

Dividing by $\delta x\ \delta y\ \delta z\ \Delta t$ and taking $\Delta t \to 0$, we find a local conservation law:

$$\text{If} \quad \frac{\partial \phi}{\partial \zeta} = \frac{\partial A_z}{\partial \zeta} = \frac{\partial A_\zeta}{\partial \zeta} = \frac{\partial A_R}{\partial \zeta} = 0 \quad \text{and collisions are negligible then} \tag{8}$$

$$\text{for each species } \alpha, \qquad \frac{\partial}{\partial t}[\Theta] + \nabla \cdot (\Theta \mathbf{u}_\alpha) = 0 \qquad \text{where} \quad \Theta = m_\alpha n_\alpha R u_{\alpha\zeta} + \frac{q_\alpha}{c} R n_\alpha A_\zeta$$

Note that the second term in the canonical angular momentum density Θ is typically large compared to the first:

$$\left(\frac{q_\alpha}{c} R n_\alpha A_\zeta \right) \Big/ \left(m_\alpha n_\alpha R u_{\alpha\zeta} \right) \sim \frac{\text{scale length for } A_\zeta}{\text{Larmor radius of species } \alpha} \gg 1 \tag{9}$$

Therefore, an approximate conservation law is

$$\text{If} \quad \frac{\partial \phi}{\partial \zeta} = \frac{\partial A_z}{\partial \zeta} = \frac{\partial A_\zeta}{\partial \zeta} = \frac{\partial A_R}{\partial \zeta} = 0 \quad \text{and collisions are negligible then} \tag{10}$$

$$\text{for each species } \alpha, \qquad \frac{\partial}{\partial t}\left[R n_\alpha A_\zeta \right] + \nabla \cdot \left(R n_\alpha A_\zeta \mathbf{u}_\alpha \right) = O\left(\rho_{L\alpha} / L \right)$$

2. Derivation from MHD equations

As one would hope, we can derive a conservation law corresponding to (10) directly from the ideal MHD equations, without making any reference to the single-particle picture:

$$\begin{aligned}
\frac{\partial}{\partial t}\left[R\rho A_\zeta\right] &= R\frac{\partial\rho}{\partial t}A_\zeta + R\rho\frac{\partial A_\zeta}{\partial t} \\
&= -RA_\zeta\nabla\cdot(\rho\mathbf{u}) + R\rho\left[-\frac{c}{R}\frac{\partial\phi}{\partial\zeta} - cE_\zeta\right] && \text{↵ mass conservation, write } \mathbf{E} \text{ in terms of potentials} \\
&= \nabla\cdot\left(-R\rho A_\zeta\mathbf{u}\right) + \rho\mathbf{u}\cdot\nabla\left(RA_\zeta\right) + R\rho\mathbf{u}\times\mathbf{B}\cdot\mathbf{e}_\zeta && \text{↵ integrate by parts, axisymmetry, Ohm's law} \\
&= \nabla\cdot\left(-R\rho A_\zeta\mathbf{u}\right) + \rho\left[u_R\frac{\partial\left(RA_\zeta\right)}{\partial R} + u_Z R\frac{\partial A_\zeta}{\partial Z}\right] + R\rho\left[u_Z B_R - u_R B_Z\right] && \text{↵ write out components} \\
&= \nabla\cdot\left(-R\rho A_\zeta\mathbf{u}\right) + \rho\left[u_R\frac{\partial\left(RA_\zeta\right)}{\partial R} + u_Z R\frac{\partial A_\zeta}{\partial Z}\right] \\
&\qquad + R\rho\left[-u_Z\frac{\partial A_\zeta}{\partial Z} - u_R\frac{1}{R}\frac{\partial}{\partial R}\left(RA_\zeta\right)\right] && \text{↵ } \mathbf{B}=\nabla\times\mathbf{A} \text{ in cylindrical geometry} \\
&= \nabla\cdot\left(-R\rho A_\zeta\mathbf{u}\right) && \text{↵ cancellation}
\end{aligned} \tag{11}$$

The result may be summarized as follows:

In ideal MHD if $\frac{\partial\phi}{\partial\zeta} = \frac{\partial A_z}{\partial\zeta} = \frac{\partial A_\zeta}{\partial\zeta} = \frac{\partial A_R}{\partial\zeta} = 0$ then $\frac{\partial}{\partial t}\left[R\rho A_\zeta\right] + \nabla\cdot\left(R\rho A_\zeta\mathbf{u}\right) = 0$ (12)

even if $\frac{\partial\rho}{\partial\zeta} \neq 0$, $\frac{\partial p}{\partial\zeta} \neq 0$, or $\frac{\partial\mathbf{u}}{\partial\zeta} \neq 0$

Note that in the MHD derivation we have used the equations of mass conservation and the ideal Ohm's law

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\mathbf{u}\right) = 0, \qquad \mathbf{E} = -\frac{1}{c}\mathbf{u}\times\mathbf{B}\,. \tag{13}$$

In the derivation (11) we have had to drop the toroidal derivatives of each one of the potential fields precisely once, so the result (12) cannot be generalized to cases in which even one of the components of $\mathbf{A}$ or ϕ is non-axisymmetric.

Interestingly, the derivation from the single-particle perspective requires that there be no collisions, whereas the second derivation applies to the high-collisionality MHD regime.