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# Dominant Balance, singular perturbations, and boundary layers

In any area of physics, but especially in plasma physics, finding exact solutions of equations is hopeless, so it is important to be able to find *approximate* solutions. One framework for doing so is the *method of dominant balance*. The idea is to first suppose that certain terms in the full equation can be ignored, so the remaining terms form the "dominant balance." If the reduced equation can be solved, then with the solution now in hand, you must go back to test your original ansatz for consistency. The process is repeated for all possible dominant balances in the full original equation.

Importantly, we will find that terms which at first appear negligible (because they include a small parameter) must in fact be kept to get the right answer!

## *Example: Problem 12.2 in Freidberg*

In problem set 6, we derived the following dispersion relation for a population of electrons moving at velocity $\upsilon$ through stationary ions, with both species at zero temperature:

$$\underbrace{\frac{1}{\omega_{pe}^2}}_{X} = \underbrace{\frac{\delta}{\omega^2}}_{Y} + \underbrace{\frac{1}{\left[\omega - \upsilon k\right]^2}}_{Z} \tag{1}$$

where $\delta = m_e / m_i$ is $= 1$. Equation (1) can be written equivalently as a quartic equation for $\omega$:

$$\left[\omega - \upsilon k\right]^2 \omega^2 = \left[\omega - \upsilon k\right]^2 \omega_{pe}^2 \delta + \omega^2 \omega_{pe}^2 \tag{2}$$

so there are evidently 4 solutions. Although an exact formula exists for solving quartic equations, it gives absolutely no insight into the character of the solutions (e.g. try solving (1) in Mathematica!)

Note that if we set $\delta = 0$ in (1), the equation becomes quadratic in $\omega$, so 2 of the 4 solutions disappear! Equation (1) is therefore said to be a **singular perturbation** problem, as opposed to a "regular perturbation" problem, because the solutions when the small parameter is nonzero are fundamentally different than the solutions when the small parameter is exactly zero. In singular perturbation problems, the "disappearing" solutions typically go to zero or $\infty$ (or become non-analytic for the case of differential equations) as the small parameter becomes zero. Equation (2) has the form of a regular perturbation problem: the polynomial remains quartic as $\delta \to 0$, so all 4 roots persist, and evidently two of the solutions go to zero.

Another important example of a singular perturbation problem in plasma physics is Bernstein waves: these waves can be found in the equations when the plasma temperature is nonzero, but they "disappear" in the cold plasma analysis in which the temperature is taken to be exactly zero.

Now let us consider the possible dominant balances in (1). There are 3 possible cases:

Case 1: $X \approx Z$.

Here we set

$$\frac{1}{\omega_{pe}^2} = \frac{1}{\left[\omega_0 - \upsilon k\right]^2} \tag{3}$$

where $\omega_0$ is our first approximation to $\omega$. Therefore $\omega_0 = \upsilon k \pm \omega_{pe}$. At least to this order, the oscillation is neither unstable nor damped. For consistency, the term $Y$ in (1) should be negligible compared to the $X$ term. Thus, the solution $\omega_0 = \upsilon k + \omega_{pe}$ is valid when $\left|\upsilon k + \omega_{pe}\right| \gg \sqrt{\delta}\omega_{pe}$, and the solution $\omega_0 = \upsilon k - \omega_{pe}$ is valid when $\left|\upsilon k - \omega_{pe}\right| \gg \sqrt{\delta}\omega_{pe}$.

<u>Case 2:</u> $X \approx Y$.

This time we set

$$\frac{1}{\omega_{pe}^2} = \frac{\delta}{\omega_0^2} \tag{4}$$

so $\omega_0 = \pm\sqrt{\delta}\omega_{pe}$. Again, at least to this order, the oscillation is neither unstable nor damped. For consistency, the term $Z$ in (1) should be negligible compared to the $X$ term. Thus, the solution $\omega_0 = +\sqrt{\delta}\omega_{pe}$ is valid when $\left|\upsilon k - \sqrt{\delta}\omega_{pe}\right| \gg \omega_{pe}$, and the solution $\omega_0 = -\sqrt{\delta}\omega_{pe}$ is valid when $\left|\upsilon k + \sqrt{\delta}\omega_{pe}\right| \gg \omega_{pe}$. Both of these consistency conditions are approximately equivalent to $\left|\upsilon k\right| \gg \omega_{pe}$.

<u>Case 3:</u> $0 \approx Y + Z$.

This time we set

$$0 = \frac{\delta}{\omega_0^2} + \frac{1}{\left[\omega - \upsilon k\right]^2} \tag{5}$$

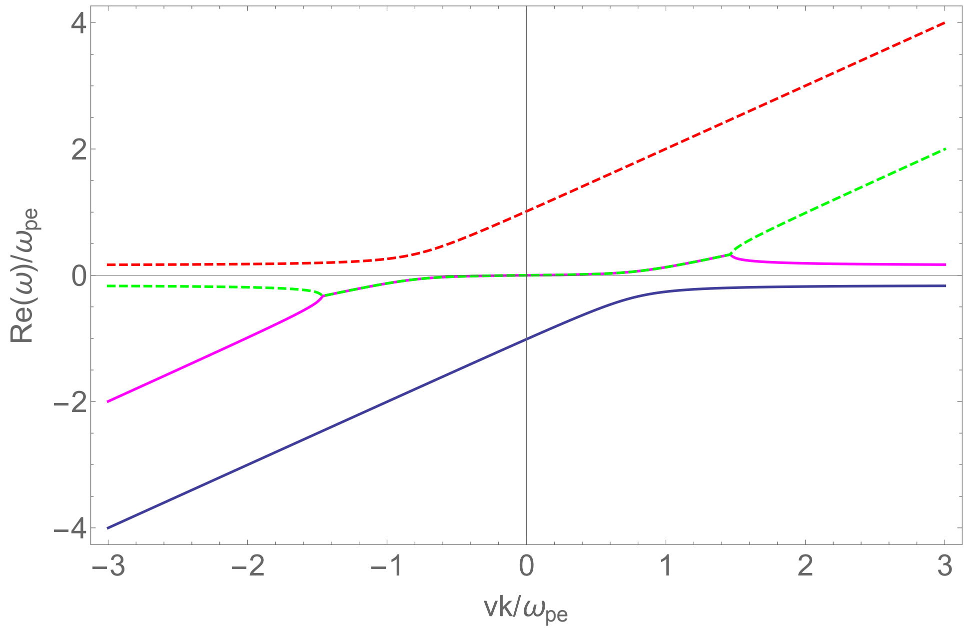
which gives the quadratic polynomial

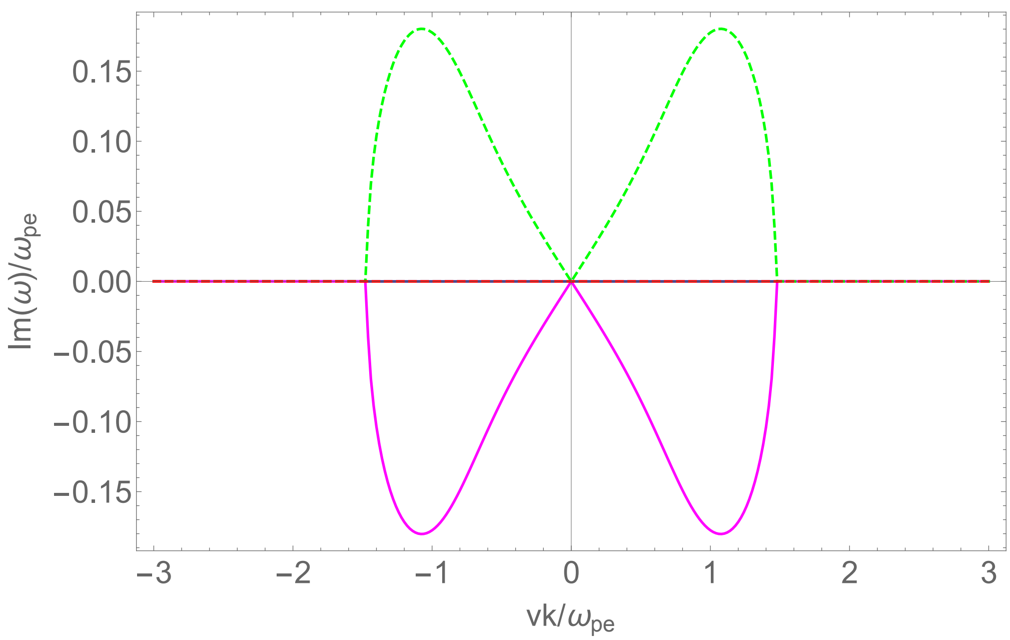
$$\left[1+\delta\right]\omega^2 - 2\delta\upsilon k\omega + \delta\upsilon^2 k^2 = 0. \tag{6}$$

We may as well drop the $\delta$ in the first term, so the solutions are $\omega_0 = \upsilon k\left[\delta \pm i\sqrt{\delta}\right]$. This time we find solutions which are almost purely growing or damped (because $\delta \ll \sqrt{\delta}$). For consistency, the term $X$ in (1) should be negligible compared to the $Y$ term. Thus, both of these solutions are valid when $\left|\upsilon k\right| \ll \omega_{pe}$. As the growth rate increases with $\left|\upsilon k\right|$, but the consistency of the calculation breaks down when $\left|\upsilon k\right| \sim \omega_{pe}$, then the most unstable $k$ must be roughly $k \sim \pm\omega_{pe}/\upsilon$.

Notice that we've found 6 solutions (2 for each of the three cases), whereas we know there can only be 4. The consistency requirements ensure that no more than 4 of our solutions are valid simultaneously.

For comparison, here are the exact solutions, found numerically for $\delta = 1/40 = 0.025$ (so $\sqrt{\delta} \approx 0.16$.) I use this unrealistically large value of $\delta$ to make the structure of the solutions easier to see on the plots. The color of each root is the same in each plot.




Notice how powerful the dominant balance method was, especially given how little effort was required! Away from $\upsilon k \approx \pm\omega_{pe}$, the dominant balance solutions give all four roots, with very accurate estimates for both the real frequency and the growth rate. The dominant balance method also provided a very good estimate for the most unstable $k$. (Close to $\upsilon k \approx \pm\omega_{pe}$, only one of our dominant balance solutions is valid: one of the Case 1 solutions.)

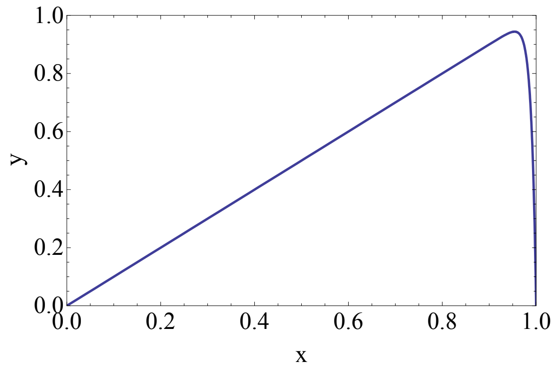
## Example 2: Differential equations and boundary layers

The method of dominant balance can be applied to essentially any type of equation, including differential equations. Consider for example

$$\underbrace{\varepsilon\frac{d^2y}{dx^2}}_{A}\underbrace{-\frac{dy}{dx}}_{B}\underbrace{+1}_{C}=0 \tag{7}$$

with $\varepsilon \ll 1$ and the boundary conditions $y(0)=0$ and $y(1)=0$. If we just set $\varepsilon=0$, we find $y=x+c$ for some constant $c$, and there is no way to satisfy the boundary conditions. Thus, this is a singular perturbation problem. The exact solution of (7) can be found to be

$$y(x)=x-\frac{\exp(x/\varepsilon)-1}{\exp(1/\varepsilon)-1}.$$




Evidently, unless $x$ is very close to 1, the dominant balance in (7) is indeed $B+C\approx 0$, giving the solution $y=x+c$ we found earlier. However, for $x$ close to 1, the dominant balance in (7) is between terms $A$ and $B$. Setting $A+B\approx 0$ we obtain $y\approx p\exp(x/\varepsilon)+q$ for some constants $p$ and $q$, which indeed describes the behavior near $x\approx 1$. This region is called a **boundary layer**. Boundary layers tend to arise when a derivative in a differential equation is multiplied by a small number.

An important instance of this type of analysis in plasma physics is reconnection, such as in a sawtooth crash or neoclassical tearing mode. In these situations, the MHD equations are a good approximation outside of a narrow boundary layer, but close to the boundary layer, more sophisticated physics models are essential. The more sophisticated physics models give rise to terms resembling term $A$ in (7): they are proportional to a small parameter, so it is tempting to drop them, but they become important when gradients are large.

## Further reading

C M Bender and S A Orszag, "Advanced Mathematical Methods for Scientists and Engineers," Springer (1999). See especially sections 7.1-7.2.

R White, "Asymptotic analysis of differential equations", Chapter 1, World Scientific (2010).
https://terpconnect.umd.edu/~mattland/plasmanotes/dominantbalance.pdf

M Kruskal, "Asymptotology", in *Mathematical models in Physical Sciences*, S Drobot ed., (1962).
https://terpconnect.umd.edu/~mattland/asymptotology.pdf