An introduction to quasisymmetry

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1 Introduction

The basic idea of quasisymmetry is the following: if the magnitude of the magnetic field $B = |\boldsymbol{B}|$ has continuous symmetry in certain special coordinate systems, in the sense that B is independent of one of the coordinates, then the guiding center particle trajectories behave exactly as if they were in a truly symmetric magnetic field. The full vector magnetic field $\boldsymbol{B}$ need not be symmetric. One reason this idea is exciting is that it means it may be possible to combine the good confinement properties of a tokamak with the stability and steady-state capability of a stellarator.

Quasisymmetry is a sufficient but not necessary condition for *omnigenity*, which is the condition that guiding center particle orbits are well confined (in the sense that their bounce averaged radial drift vanishes.)

Quasisymmetry can be motivated and explored from several perspectives, since it has many implications. A number of these perspectives will be explored in this note, but not all.

2 Boozer coordinates

To explain some aspects of quasisymmetry, it is necessary to understand Boozer coordinates. Boozer coordinates are (s, θ, ζ) where s is some flux surface label, and θ and ζ are poloidal and toroidal angles defined in a particular way. For concreteness, we will use $s = \psi$, where $2\pi\psi$ is the poloidal flux, though this choice is not necessary. The three coordinates are generally not orthogonal.

2.1 General non-orthogonal coordinates

For non-orthogonal coordinates, there are two natural sets of basis vectors that can be used to represent general vector quantities. One set of basis vectors is $(\nabla\psi,\ \nabla\theta,\ \nabla\zeta)$, where

$$\nabla\psi = \boldsymbol{e}_x \frac{\partial\psi}{\partial x} + \boldsymbol{e}_y \frac{\partial\psi}{\partial y} + \boldsymbol{e}_z \frac{\partial\psi}{\partial z} \tag{1}$$

with analogous definitions for $\nabla\theta$ and $\nabla\zeta$. Here, y and z are held fixed in the $\partial/\partial x$ differentiation, the analogous statements hold for $\partial/\partial y$ and $\partial/\partial z$, and $\boldsymbol{e}_{x,y,z}$ are the Cartesian unit basis vectors. The other set of basis vectors is $(\partial\boldsymbol{r}/\partial\psi,\ \partial\boldsymbol{r}/\partial\theta,\ \partial\boldsymbol{r}/\partial\zeta)$, where

$$\frac{\partial\boldsymbol{r}}{\partial\psi} = \boldsymbol{e}_x \frac{\partial x}{\partial\psi} + \boldsymbol{e}_y \frac{\partial y}{\partial\psi} + \boldsymbol{e}_z \frac{\partial z}{\partial\psi}, \tag{2}$$

with analogous definitions for $\partial\boldsymbol{r}/\partial\theta$ and $\partial\boldsymbol{r}/\partial\zeta$. Here, in contrast to (1), θ and ζ are held fixed in the $\partial/\partial\psi$ differentiation, and analogous statements hold for $\partial/\partial\theta$ and $\partial/\partial/\zeta$. If the coordinates were orthogonal, then $\nabla\psi$ would be parallel to $\partial\boldsymbol{r}/\partial\psi$, and similarly for the other two coordinates.

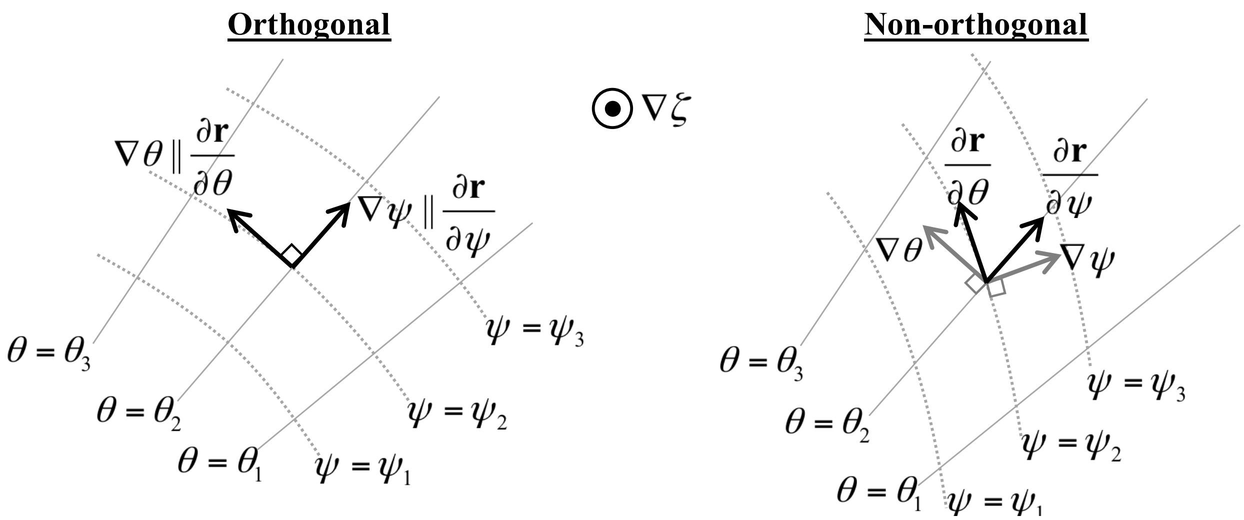

Figure 1: Orthogonal vs non-orthogonal coordinates.

However, for non-orthogonal coordinates these directions are generally different. This important difference between orthogonal vs non-orthogonal coordinates is illustrated in figure 1.

For any vector $\boldsymbol{B}$, (not only the magnetic field), we can decompose the vector in either basis, writing

$$\boldsymbol{B} = B_\psi \nabla\psi + B_\theta \nabla\theta + B_\zeta \nabla\zeta \tag{3}$$

and

$$\boldsymbol{B} = B^\psi \frac{\partial \boldsymbol{r}}{\partial\psi} + B^\theta \frac{\partial \boldsymbol{r}}{\partial\theta} + B^\zeta \frac{\partial \boldsymbol{r}}{\partial\zeta}. \tag{4}$$

We can relate the two sets of basis vectors as follows. First, note that the vector $\partial\boldsymbol{r}/\partial\psi$ by definition points in a direction along which θ and ζ do not increase. Thus,

$$\frac{\partial \boldsymbol{r}}{\partial\psi}\cdot\nabla\theta = 0 \quad \text{and} \quad \frac{\partial \boldsymbol{r}}{\partial\psi}\cdot\nabla\zeta = 0. \tag{5}$$

It follows that

$$\frac{\partial \boldsymbol{r}}{\partial\psi} = J\nabla\theta\times\nabla\zeta \tag{6}$$

for some coefficient J. To determine this coefficient, consider a step $d\boldsymbol{r}$ at fixed θ and ζ: $d\psi = d\boldsymbol{r}\cdot\nabla\psi$, so $(\partial\boldsymbol{r}/\partial\psi)\cdot\nabla\psi = 1$. (This same result can also be seen by forming the dot product of (1) with (2) and recognizing the result as the chain rule applied to $\partial\psi(\boldsymbol{r})/\partial\psi = 1$, where $\boldsymbol{r} = \boldsymbol{r}(x, y, z)$.) The dot product of $\nabla\psi$ with (6) therefore tells us

$$J = \frac{1}{\nabla\psi\cdot\nabla\theta\times\nabla\zeta}. \tag{7}$$

By repeating the same analysis with cyclic permutations of the coordinates, we arrive at the so-called 'dual relations':

$$\frac{\partial \boldsymbol{r}}{\partial\psi} = J\nabla\theta\times\nabla\zeta, \qquad \frac{\partial \boldsymbol{r}}{\partial\theta} = J\nabla\zeta\times\nabla\psi, \qquad \frac{\partial \boldsymbol{r}}{\partial\zeta} = J\nabla\psi\times\nabla\theta. \tag{8}$$

Therefore, an equivalent expression to (4) is

$$\boldsymbol{B} = \frac{1}{J}\left(B^\psi\nabla\theta\times\nabla\zeta + B^\theta\nabla\zeta\times\nabla\psi + B^\zeta\nabla\psi\times\nabla\theta\right). \tag{9}$$

Now let us specialize to the case in which $\boldsymbol{B}$ is the magnetic field. If good magnetic surfaces exist, then $\boldsymbol{B}\cdot\nabla\psi = 0$. Applying $\cdot\nabla\psi$ to (9), then, we find $B^\psi = 0$. There are consequently 5 nonzero components of $\boldsymbol{B}$ between the two representations: $(B_\psi, B_\theta, B_\zeta, B^\theta, B^\zeta)$.

2.2 Boozer coordinates

There are many choices which can be made for θ and ζ. One choice for ζ is the standard cylindrical azimuthal angle, and one choice for θ is $\text{atan2}(z - z_0,\, R - R_0)$ where atan2 is the arctangent with appropriate sign, $R = \sqrt{x^2 + y^2}$ is the cylindrical radius (i.e. major radius), and $(z_0(\zeta),\, R_0(\zeta))$ are the location of the magnetic axis. However, we can add any single-valued functions to one set of angles to get another set of angle coordinates:

$$\theta' = \theta + f(\boldsymbol{r}), \qquad\qquad \zeta' = \zeta + g(\boldsymbol{r}) \tag{10}$$

It can be shown that this freedom can be used to make two of the components of $\boldsymbol{B}$ simplify:

$$B^\theta/J = 1, \qquad\qquad B^\zeta/J = q, \tag{11}$$

where $q(\psi)$ is the safety factor, which is constant on a magnetic surface. The details of this construction will not be given here, but they can be found in section 2.2 of [1]. Then using the result

$$\frac{d\psi_t}{d\psi} = q \tag{12}$$

where $2\pi\psi_t$ is the toroidal flux, then (9) becomes

$$\begin{aligned}\boldsymbol{B} &= \nabla\zeta\times\nabla\psi + q\nabla\psi\times\nabla\theta \\ &= \nabla\zeta\times\nabla\psi + \nabla\psi_t\times\nabla\theta.\end{aligned} \tag{13}$$

Coordinates in which (13) is satisfied are "straight field line coordinates".

There is still freedom left in the transformation (10), and this freedom can be used to simplify other components of $\boldsymbol{B}$. Boozer coordinates are constructed by using the transformation (10) to make B_θ and B_ζ constant on magnetic surfaces:

$$\boldsymbol{B} = B_\psi(\psi,\theta,\zeta)\nabla\psi + B_\theta(\psi)\nabla\theta + B_\zeta(\psi)\nabla\zeta. \tag{14}$$

We will not give the construction here; it is given in section 2.5 of [1].

Taking the dot product of (13) with (14), we obtain a useful expression:

$$\nabla\psi\cdot\nabla\theta\times\nabla\zeta = \frac{B^2}{qB_\zeta(\psi) + B_\theta(\psi)}. \tag{15}$$

This quantity is the (inverse) Jacobian of the transformation between Cartesian and Boozer coordinates. For understanding quasisymmetry, it is noteworthy that this Jacobian varies on a magnetic surface only through B.

2.3 The radial Boozer component

A result we will need below is the following: a symmetry of B in Boozer angles implies that B_ψ has the same symmetry. This result is obtained by considering the MHD equilibrium relation

$$\boldsymbol{j}\times\boldsymbol{B} = \nabla p, \tag{16}$$

where $\mu_0 \boldsymbol{j} = \nabla \times \boldsymbol{B}$, $\boldsymbol{j}$ is the current density, and $p = p(\psi)$ is the plasma pressure. We evaluate $\boldsymbol{j}$ from the curl of (14):

$$\nabla \times \boldsymbol{B} = \nabla B_\psi \times \nabla\psi + \frac{dB_\theta}{d\psi}\nabla\psi \times \nabla\theta + \frac{dB_\zeta}{d\psi}\nabla\psi \times \nabla\zeta. \tag{17}$$

In the first right-hand side term, note that

$$\nabla B_\psi = \frac{\partial B_\psi}{\partial\psi}\nabla\psi + \frac{\partial B_\psi}{\partial\theta}\nabla\theta + \frac{\partial B_\psi}{\partial\zeta}\nabla\zeta. \tag{18}$$

Forming (17) $\times$ (13), one finds the MHD equilibrium equation (16) becomes

$$\left(q\frac{\partial B_\psi}{\partial\zeta} + \frac{\partial B_\psi}{\partial\theta} - q\frac{dB_\zeta}{d\psi} - \frac{dB_\theta}{d\psi}\right)(\nabla\psi \cdot \nabla\theta \times \nabla\zeta)\,\nabla\psi = \mu_0\frac{dp}{d\psi}\nabla\psi. \tag{19}$$

The directions of the right and left sides of this equation are always parallel, so we can remove the $\nabla\psi$ vectors from each side. Applying (15),

$$q\frac{\partial B_\psi}{\partial\zeta} + \frac{\partial B_\psi}{\partial\theta} - q\frac{dB_\zeta}{d\psi} - \frac{dB_\theta}{d\psi} = \mu_0\frac{qB_\zeta + B_\theta}{B^2}\frac{dp}{d\psi}. \tag{20}$$

Writing Fourier expansions for $1/B^2$ and B_ψ,

$$\begin{aligned} \frac{1}{B^2} &= \sum_{m,n}\mathcal{B}_{m,n}e^{im\theta - in\zeta}, \\ B_\psi &= \sum_{m,n}B_{\psi,m,n}e^{im\theta - in\zeta}, \end{aligned} \tag{21}$$

then (20) gives

$$B_{\psi,m,n} = \frac{\mu_0\mathcal{B}_{m,n}(qB_\zeta + B_\theta)}{i(m - qn)}\frac{dp}{d\psi}. \tag{22}$$

(Note that the $m = n = 0$ term of B_ψ is not determined by (20).) Therefore, if B is independent of ζ, so all the $n \neq 0$ terms in $\mathcal{B}_{m,n}$ vanish, then all the $n \neq 0$ terms of $B_{\psi,m,n}$ will vanish as well, so B_ψ will be independent of ζ.

More generally, suppose B depends on θ and ζ only through the helical combination

$$\chi = M\theta - N\zeta \tag{23}$$

where M and N and fixed integers. Then this same property is true of $1/B^2$. Hence, $\mathcal{B}_{m,n}$ vanishes unless $m = kM$ and $n = kN$ for some rational number k. By (22), then $B_{\psi,m,n}$ vanishes unless $m = kM$ and $n = kN$. Hence, B_ψ depends on θ and ζ only through the combination χ, exactly like B.

3 The guiding-center Lagrangian

One perspective on quasisymmetry comes from the Lagrangian for guiding center motion. We will show that in Boozer coordinates, this Lagrangian depends on position only through the radial (flux surface label) coordinate and through B. There are two important consequences. First, the guiding-center dynamics of two plasmas with the same B and radial profiles are isomorphic. Second, if B

can be made independent of one of the Boozer angles (or a linear combination thereof), there will be a conserved quantity by Noether's theorem. This conserved quantity ensures good confinement of the guiding centers.

The Lagrangian for guiding center motion has been derived by Littlejohn [2], and is

$$L(\boldsymbol{r}, v_{||}, \mu, \varphi, \dot{\boldsymbol{r}}, \dot{v}_{||}, \dot{\mu}, \dot{\varphi}, t) = Ze\boldsymbol{A}\cdot\dot{\boldsymbol{r}} + mv_{||}\boldsymbol{b}\cdot\dot{\boldsymbol{r}} + \frac{m}{Ze}\mu\dot{\varphi} - \frac{mv_{||}^2}{2} - \mu B - Ze\Phi, \tag{24}$$

where a dot on top of a symbol denotes a time derivative. Here, $\boldsymbol{b} = \boldsymbol{B}/B$ is a unit vector, Ze is the particle charge, μ is the magnetic moment, φ is the gyrophase, $\boldsymbol{A}$ is the vector potential, and Φ is the electrostatic potential. Forming the Euler-Lagrange equations for this Lagrangian, one finds the following equations of motion:

$$\dot{\boldsymbol{r}} = v_{||}\boldsymbol{b} + \frac{1}{ZeB_{||}^*}\boldsymbol{b}\times(mv_{||}^2\boldsymbol{b}\cdot\nabla\boldsymbol{b} + \mu\nabla\boldsymbol{b} - Ze\boldsymbol{E}^*), \tag{25}$$

$$\dot{v}_{||} = -\frac{1}{m}\boldsymbol{b}\cdot(\mu\nabla B - Ze\boldsymbol{E}) + v_{||}\boldsymbol{b}\cdot\nabla\boldsymbol{b}\cdot\dot{\boldsymbol{r}}, \tag{26}$$

$$\dot{\varphi} = ZeB/m, \tag{27}$$

$$\dot{\mu} = 0, \tag{28}$$

where m is the particle mass, Ze is the particle charge, $\boldsymbol{E}^* = -\nabla\Phi - \partial\boldsymbol{A}^*/\partial t$, $B_{||}^* = \boldsymbol{B}^*\cdot\boldsymbol{b}$, $\boldsymbol{B}^* = \nabla\times\boldsymbol{A}^*$, and $\boldsymbol{A}^* = \boldsymbol{A} + m(Ze)^{-1}v_{||}\boldsymbol{b}$. To leading order in $\rho_* = v_{||}/(\Omega L)$ where $\Omega = ZeB/m$ and L is a typical scale length, then $B_{||}^* \approx B$. Note also that the last term in (26) is small compared to the first right-hand-side term by a factor of ρ_*, since $\boldsymbol{b}\cdot\nabla\boldsymbol{b}\cdot\boldsymbol{b} = 0$. Hence, (25)-(28) give the usual guiding center equations of motion. The Lagrangian for guiding-center motion is also discussed in section 6.3 of [3].

Let us now write the Lagrangian (24) in Boozer coordinates. We will assume $\boldsymbol{B}$ is time-independent for simplicity. We need an expression for $\boldsymbol{A}$ in some gauge, and a suitable expression is

$$\boldsymbol{A} = \psi_t\nabla\theta - \psi\nabla\zeta. \tag{29}$$

It can be seen that the curl of this expression is (13). The Lagrangian (24) also includes $\dot{\boldsymbol{r}}$, which can be written

$$\dot{\boldsymbol{r}} = \dot{\psi}\frac{\partial\boldsymbol{r}}{\partial\psi} + \dot{\theta}\frac{\partial\boldsymbol{r}}{\partial\theta} + \dot{\zeta}\frac{\partial\boldsymbol{r}}{\partial\zeta} \tag{30}$$

$$= \frac{1}{J}\left(\dot{\psi}\nabla\theta\times\nabla\zeta + \dot{\theta}\nabla\zeta\times\nabla\psi + \dot{\zeta}\nabla\psi\times\nabla\theta\right), \tag{31}$$

where we have used the dual relations. Substituting (29)-(31) into the Lagrangian (24), we obtain

$$L(\boldsymbol{r}, v_{||}, \mu, \varphi, \dot{\boldsymbol{r}}, \dot{v}_{||}, \dot{\mu}, \dot{\varphi}) = Ze\psi_t\dot{\theta} - Ze\psi\dot{\zeta} + \frac{mv_{||}}{B}\left(\dot{\psi}B_\psi + \dot{\theta}B_\theta + \dot{\zeta}B_\zeta\right) + \frac{m}{Ze}\mu\dot{\varphi} - \frac{mv_{||}^2}{2} - \mu B - Ze\Phi. \tag{32}$$

It can be seen that this Lagrangian depends on position only through (1) the flux functions ψ, ψ_t, B_θ, and B_ζ, (2) the field magnitude B, and (3) the electrostatic potential Φ. Thus if B and Φ are independent of ζ, the particle dynamics will be exactly analogous to the dynamics in a tokamak. If B is independent of ζ, the mean Φ will tend to be independent of ζ as well since Φ is a density moment of the guiding-center distribution function. Hence, the requirement that Φ have symmetry is not a major restriction.

An ignorable coordinate can also arise in the case of helical symmetry. To define this kind of symmetry, we introduce

$$\chi = M\theta - N\zeta \tag{33}$$

where M and N and fixed integers. Noting $\dot{\chi} = M\dot{\theta} - N\dot{\zeta}$, (32) is equivalent to

$$L = \frac{Ze\psi_t}{M}\left(\dot{\chi} + N\dot{\zeta}\right) - Ze\psi\dot{\zeta} + \frac{mv_{||}}{B}\left[\dot{\psi}B_\psi + \frac{B_\theta}{M}\left(\dot{\chi} + N\dot{\zeta}\right) + \dot{\zeta}B_\zeta\right] + \frac{m}{Ze}\mu\dot{\varphi} - \frac{mv_{||}^2}{2} - \mu B - Ze\Phi. \tag{34}$$

If $B = B(\psi, \chi)$, i.e. if B depends on θ and ζ only through the combination χ, then (34) looks identical to the Lagrangian (32) for an axisymmetric magnetic field $B(\psi, \theta)$, but with certain substitutions made.

4 Conserved canonical angular momentum and Tamm's theorem

4.1 Noether's theorem

Noether's theorem is the observation that for any quantity Q and Lagrangian L, if $\partial L/\partial Q = 0$ (i.e. Q is an ignorable coordinate), then the Euler-Lagrange equation is $(d/dt)\partial L/\partial \dot{Q} = 0$, so $\partial L/\partial \dot{Q}$ is a conserved quantity. From (32), we see that in the case of quasi-axisymmetry $B = B(\psi, \theta)$, the conserved quantity is

$$\left(\frac{\partial L}{\partial \dot{\zeta}}\right)_{\theta,\dot{\theta},\zeta} = -Ze\psi + \frac{mv_{||}B_\zeta}{B}. \tag{35}$$

For helical symmetry, where the Lagrangian is independent of ζ at fixed χ, the conserved quantity obtained from (34) is

$$\left(\frac{\partial L}{\partial \dot{\zeta}}\right)_{\chi,\dot{\chi},\zeta} = Ze\frac{N}{M}\psi_t - Ze\psi + \frac{mv_{||}}{B}\left(B_\zeta + \frac{N}{M}B_\theta\right). \tag{36}$$

4.2 Comparison with true axisymmetry

These conserved quantities for quasisymmetric magnetic fields resemble the canonical angular momentum

$$p_\phi = R\left[m\boldsymbol{v} + Ze\boldsymbol{A}\right]\cdot \boldsymbol{e}_\phi \tag{37}$$

which is conserved in a truly axisymmetric magnetic field. Here, R is the cylindrical radius, and $\boldsymbol{e}_\phi$ is the unit vector in the direction of the standard azimuthal angle of cylindrical coordinates, ϕ. Note that ϕ differs from the toroidal Boozer angle ζ even in axisymmetry. Eq (37) is the conserved quantity associated with Noether's Theorem for the *full* single-particle Lagrangian, as opposed to the guiding center Lagrangian.

To understand the correspondence in detail, we need to use the result that truly axisymmetric magnetic fields can be represented

$$\boldsymbol{B} = \nabla\phi \times \nabla\psi + B_\zeta(\psi)\nabla\phi, \tag{38}$$

where $\nabla\phi = R^{-1}\boldsymbol{e}_\phi$, and $B_\zeta(\psi)$ is the same quantity appearing in the Boozer coordinate representation (14). The fact that the same quantity B_ζ appears in both (14) and (38) can be seen by applying Ampere's Law to a loop in the toroidal direction. At the same time, from the standard formula for the curl in cylindrical coordinates, we find that in any gauge in which $\boldsymbol{A}$ is axisymmetric (so $\partial A_R/\partial\phi = 0$ and $\partial A_Z/\partial\phi = 0$),

$$\boldsymbol{B} = -\boldsymbol{e}_R \frac{1}{R}\frac{\partial(RA_\phi)}{\partial Z} + \boldsymbol{e}_\phi \left(\frac{\partial A_R}{\partial Z} - \frac{\partial A_Z}{\partial R}\right) + \boldsymbol{e}_Z \frac{1}{R}\frac{\partial(RA_\phi)}{\partial R}. \tag{39}$$

Comparing this expression with (38), we find the R and Z components imply $RA_\phi = -\psi +$ constant. Thus the $\boldsymbol{A}$ term in (37) is identical to the ψ term in (35) (up to an unimportant constant.)

The correspondence between the two conservation laws is still not obviously complete, since (37) involves v_ϕ whereas (35) involves $v_{||}$. This difference can be understood in terms of gyroaveraging, as follows. Squaring (38) yields $B^2 = (|\nabla\psi|^2 + B_\zeta^2)/R^2$. Applying the result to (38)$\times\nabla\psi$, one finds

$$\boldsymbol{B} \times \nabla\psi = B_\zeta \boldsymbol{B} - R^2 B^2 \nabla\phi. \tag{40}$$

Therefore, the velocity term in (37) can be written

$$Rm\boldsymbol{v}\cdot\boldsymbol{e}_\phi = R^2 m\boldsymbol{v}\cdot\nabla\phi = \frac{m}{B}\boldsymbol{v}\cdot(B_\zeta \boldsymbol{b} - \boldsymbol{b}\times\nabla\psi). \tag{41}$$

The $\boldsymbol{v}\cdot\boldsymbol{b}$ term in (41) gives the $v_{||}$ term in (35), while the $\boldsymbol{v}\cdot\boldsymbol{b}\times\nabla\psi$ term in (41) averages to zero over the Larmor gyration. Thus, (35) corresponds precisely to the gyroaverage of (37) (up to a constant.) This correspondence make sense, since the former applies to the gyrocenter while the latter applies to the exact particle position.

4.3 Tamm's theorem

A very important consequence of the conserved quantity (36) is that particle trajectories are well confined, at least in the absence of collisions and turbulence. The argument, which has been known for true axisymmetry under the name Tamm's Theorem, goes as follows. The v terms in the conservation law are small compared to the ψ terms by a factor $\sim \rho_*$, which is $\leq 1/100$ for thermal particles in strongly magnetized laboratory plasmas. Thus, to $\mathcal{O}(\rho_*)$, ψ is conserved along particle trajectories. Thus, particles stick to flux surfaces to within a factor of $\mathcal{O}(\rho_*)$ even if they are trapped.

4.4 Direct calculation of conservation law

The conservation of canonical angular momentum in quasisymmetry can also be shown in a direct way, without reference to a Lagrangian. Such a proof for the case of general helicity is given in the appendix of [4] and in appendix A of [5]. Here we consider just the quasi-axisymmetry case for simplicity. Our goal is to show that (35) is constant along the guiding center trajectory, using $\partial B/\partial\zeta = 0$ but not assuming true axisymmetry. More precisely, the equation we wish to derive is

$$\left(v_{||}\boldsymbol{b} + \boldsymbol{v}_d\right)\cdot\nabla\left(-Ze\psi + \frac{mv_{||}B_\zeta}{B}\right) = 0, \tag{42}$$

where the gradient is taken at fixed μ and total energy

$$W = \frac{mv_{||}^2}{2} + \mu B + \frac{Ze}{m}\Phi, \tag{43}$$

where $\boldsymbol{v}_d$ is the cross-field drift. We use the following expression for this drift:

$$\boldsymbol{v}_d = \frac{mv_{||}}{ZeB}\nabla\times(v_{||}\boldsymbol{b}), \tag{44}$$

where again the gradient is performed at fixed μ and W.

It can be seen that there are four terms in (42). One term vanishes by itself:

$$v_{||}\boldsymbol{b}\cdot\nabla(-Ze\psi) = 0. \tag{45}$$

Another term is

$$\begin{aligned}
\boldsymbol{v}_d\cdot\nabla(-Ze\psi) =& -Ze\frac{mv_{||}}{ZeB}\nabla\times(v_{||}\boldsymbol{b})\cdot\nabla\psi \\
=& -\frac{mv_{||}}{B}\nabla\times\left[\frac{v_{||}}{B}(B_\psi\nabla\psi + B_\theta\nabla\theta + B_\zeta\nabla\zeta)\right]\cdot\nabla\psi \\
=& -\frac{mv_{||}}{B}\left[\nabla\left(\frac{v_{||}B_\psi}{B}\right)\times\nabla\psi\cdot\nabla\psi + \nabla\left(\frac{v_{||}B_\theta}{B}\right)\times\nabla\theta\cdot\nabla\psi + \nabla\left(\frac{v_{||}B_\zeta}{B}\right)\times\nabla\zeta\cdot\nabla\psi\right] \\
=& -\frac{mv_{||}}{B}\left[(\nabla\zeta\times\nabla\theta\cdot\nabla\psi)B_\theta\frac{\partial}{\partial\zeta}\left(\frac{v_{||}}{B}\right) + (\nabla\theta\times\nabla\zeta\cdot\nabla\psi)B_\zeta\frac{\partial}{\partial\theta}\left(\frac{v_{||}}{B}\right)\right]
\end{aligned} \tag{46}$$

where we have applied (14) and (44). In the last line, the quantity $\partial(v_{||}/B)/\partial\zeta$ (at fixed W and μ) vanishes in quasi-axisymmetry, since $\partial B/\partial\zeta = 0$, and since $v_{||}$ depends on position only through B in light of (43). Then noting from (13) that $\boldsymbol{B}\cdot\nabla\theta = \nabla\zeta\times\nabla\psi\times\nabla\theta$, (46) reduces to

$$\begin{aligned}
\boldsymbol{v}_d\cdot\nabla(-Ze\psi) =& -\frac{mv_{||}}{B}(\nabla\theta\times\nabla\zeta\cdot\nabla\psi)B_\zeta\frac{\partial}{\partial\theta}\left(\frac{v_{||}}{B}\right) \\
=& -\frac{mv_{||}}{B}(\boldsymbol{B}\cdot\nabla\theta)\frac{\partial}{\partial\theta}\left(\frac{v_{||}B_\zeta}{B}\right) = -v_{||}\boldsymbol{b}\cdot\nabla\left(\frac{mv_{||}B_\zeta}{B}\right).
\end{aligned} \tag{47}$$

The last term is equal and opposite to one of the other terms in (42). Thus, we have shown

$$(v_{||}\boldsymbol{b}+\boldsymbol{v}_d)\cdot\nabla\left(-Ze\psi + \frac{mv_{||}B_\zeta}{B}\right) = \boldsymbol{v}_d\cdot\nabla\left(\frac{mv_{||}B_\zeta}{B}\right). \tag{48}$$

It remains to show that this last term vanishes, which can be done as follows. Applying (44),

$$\begin{aligned}
\boldsymbol{v}_d\cdot\nabla\left(\frac{mv_{||}B_\zeta}{B}\right) =& \frac{m^2v_{||}}{ZeB}\nabla\times\left(\frac{v_{||}}{B}\boldsymbol{B}\right)\cdot\nabla\left(\frac{v_{||}B_\zeta}{B}\right) \\
=& \frac{m^2v_{||}}{ZeB}\left[\left(\nabla\frac{v_{||}}{B}\right)\times\boldsymbol{B} + \frac{v_{||}}{B}\nabla\times\boldsymbol{B}\right]\cdot\left[B_\zeta\nabla\frac{v_{||}}{B} + \frac{v_{||}}{B}\frac{dB_\zeta}{d\psi}\nabla\psi\right] \\
=& \frac{m^2v_{||}}{ZeB}\left[\left(\nabla\frac{v_{||}}{B}\right)\times\boldsymbol{B}\cdot\left(\frac{v_{||}}{B}\frac{dB_\zeta}{d\psi}\nabla\psi\right) + \frac{v_{||}}{B}\nabla\times\boldsymbol{B}\cdot\left(B_\zeta\nabla\frac{v_{||}}{B}\right)\right] \\
=& \frac{m^2v_{||}}{ZeB}\left[\frac{\partial}{\partial\theta}\left(\frac{v_{||}}{B}\right)\right]\frac{v_{||}}{B}\left[\frac{dB_\zeta}{d\psi}\nabla\theta\times\boldsymbol{B}\cdot\nabla\psi + B_\zeta\nabla\times\boldsymbol{B}\cdot\nabla\theta\right] \\
=& \frac{m^2v_{||}}{ZeB}\left[\frac{\partial}{\partial\theta}\left(\frac{v_{||}}{B}\right)\right]\frac{v_{||}}{B}\left\{\frac{dB_\zeta}{d\psi}B_\zeta\nabla\theta\times\nabla\zeta\cdot\nabla\psi + B_\zeta\nabla\times\boldsymbol{B}\cdot\nabla\theta\right\}.
\end{aligned} \tag{49}$$

In these steps, we have used the MHD equilibrium relation $\nabla\times\boldsymbol{B}\cdot\nabla\psi = 0$. In the last term of (49) we substitute (17) and note that $\partial B_\psi/\partial\zeta$ due to the analysis of section 2.3. Therefore, the terms in curly braces in the last line of (49) sum to 0, and so the right-hand side of (48) vanishes, proving the conservation equation (42).

The direct proof of the conserved helical momentum in the case of helical symmetry is exactly analogous.

5 Neoclassical transport

Another perspective on quasisymmetry comes from considering the calculation of neoclassical transport (radial fluxes and parallel flows, including the parallel current.) We will see that in a quasisymmetric magnetic field, this calculation yields identical results to the calculation in a tokamak, up to a few substitutions. Consequently, the large $1/\nu$ transport regime which is problematic for general stellarators is absent.

Neoclassical transport is calculated by solving the drift-kinetic equation

$$v_{||}\boldsymbol{b}\cdot\nabla f_1 + (\boldsymbol{v}_d\cdot\nabla\psi)\frac{\partial f_M}{\partial\psi} = C[f_1] \tag{50}$$

where

$$f_M = n(\psi)\left[\frac{m}{2\pi T(\psi)}\right]^{3/2}\exp\left(-\frac{mv^2}{2T(\psi)}\right) \tag{51}$$

is the leading-order Maxwellian flux function, f_1 is the correction to the Maxwellian, C is the linearized collision operator, and gradients are performed at fixed μ and W. The potential Φ can be taken to be a flux function to leading order, so v is constant on magnetic surfaces. Once the drift-kinetic equation is solved for f_1, the particle fluxes are computed from

$$\Gamma = \left\langle \int d^3v\, f_1\boldsymbol{v}_d\cdot\nabla\psi\right\rangle, \tag{52}$$

where $\langle\ldots\rangle$ denotes a flux surface average, the same integral with an extra factor of energy gives the energy flux, and the parallel flows are computed from

$$nV_{||} = \int d^3v\, f_1 v_{||}. \tag{53}$$

Now let us consider each term in (50)-(53) in Boozer coordinates, showing that each depends on the geometry only through ψ and B. This is already clear for $v_{||}$ due to (43). The rest of the streaming term in (50) is

$$\begin{aligned}\boldsymbol{b}\cdot\nabla f_1 =& \frac{1}{B}\left[(\boldsymbol{B}\cdot\nabla\theta)\frac{\partial f_1}{\partial\theta} + (\boldsymbol{B}\cdot\nabla\zeta)\frac{\partial f_1}{\partial\zeta}\right]\\ =& \frac{1}{B}(\nabla\psi\cdot\nabla\theta\times\nabla\zeta)\left[\frac{\partial f_1}{\partial\theta} + q\frac{\partial f_1}{\partial\zeta}\right]\\ =& \frac{B}{qB_\zeta + B_\theta}\left[\frac{\partial f_1}{\partial\theta} + q\frac{\partial f_1}{\partial\zeta}\right],\end{aligned} \tag{54}$$

where we have used (15). Thus, the streaming term introduces dependence on geometry only through ψ and B.

Next, the radial drift term, from (46) and (15), is

$$\boldsymbol{v}_d\cdot\nabla\psi = \frac{mv_{||}}{ZeB}\frac{B^2}{qB_\zeta + B_\theta}\left[B_\zeta\frac{\partial}{\partial\theta}\left(\frac{v_{||}}{B}\right) - B_\theta\frac{\partial}{\partial\zeta}\left(\frac{v_{||}}{B}\right)\right]. \tag{55}$$

In this form it can be seen that this term depends on position only through ψ and B.

Finally, the linearized collision operator C can be expressed in terms of $v_{||}$ and $v_\perp$, so it introduces no additional geometry dependence. Thus, if B varies on a ψ surface only through θ, i.e. it is independent of ζ as in true axisymmetry, the solution f_1 will have the same property (at fixed

μ). Or in the case of helical symmetry, where B varies on a ψ surface only through χ, then f_1 will do so as well.

The last step is to show that the integrals (52)-(53) introduce no additional geometry dependence. For the latter, it is sufficient to observe

$$\int d^3v = 2\pi \sum_\sigma \int_0^{v^2/(2B)} d\mu \int_0^\infty dv \frac{Bv}{|v_{||}|} \tag{56}$$

where $\sigma = \mathrm{sgn}(v_{||})$, which depends on geometry only through ψ and B. To show that (52) does not introduce extra geometry dependence, we note that the flux surface average of any quantity Q is

$$\langle Q \rangle = \frac{\int_0^{2\pi} d\theta \int_0^{2\pi} d\zeta \frac{Q}{\nabla\psi\cdot\nabla\theta\times\nabla\zeta}}{\int_0^{2\pi} d\theta \int_0^{2\pi} d\zeta \frac{1}{\nabla\psi\cdot\nabla\theta\times\nabla\zeta}} = \frac{\int_0^{2\pi} d\theta \int_0^{2\pi} d\zeta \frac{Q}{B^2}}{\int_0^{2\pi} d\theta \int_0^{2\pi} d\zeta \frac{1}{B^2}}. \tag{57}$$

Hence the flux surface average does not introduce geometry dependence beyond ψ and B. Therefore the same is true of the integral in (52).

Thus, in all the steps for calculating neoclassical phenomena, the equations 'know' about the geometry only through the radial coordinate and through B. A symmetry of B therefore causes the calculation to be isomorphic to the calculation in a tokamak.

6 Equivalent conditions for quasisymmetry

There are in fact several equivalent conditions for quasisymmetry. The following statements are equivalent:

1. $B = B(\psi, M\theta - N\zeta)$ for Boozer angles θ and ζ.
2. $B = B(\psi, M\theta' - N\zeta')$ for angles θ' and ζ' in any other coordinate system for which the Jacobian $(\nabla\psi \cdot \nabla\theta' \times \nabla\zeta')^{-1}$ depends on position only through ψ and B.
3. $\boldsymbol{B} \times \nabla\psi \cdot \nabla B = F(\psi)\boldsymbol{B} \cdot \nabla B$ for some flux function $F(\psi)$.
4. $\nabla B \times \nabla\psi \cdot \nabla(\boldsymbol{B} \cdot \nabla B) = 0$.
5. $B = B(\psi, \ell)$ where ℓ is the arclength along the field.

The only common coordinate system to which condition (2) applies is Hamada coordinates. The equivalence of (1) and (2) is shown in appendix A of [6], appendix B of [7], and page 20 of [1].

Finish

7 Possible helicities

impossibility of quasi-poloidal...

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