The H Theorem for the Landau-Fokker-Planck Collision Operator

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In these notes, we will consider the (nonlinear) Landau-Fokker-Planck collision operator for Coulomb collisions between particles of a single species, and we will prove the following two results:

1. The entropy production rate is non-negative.

2. The entropy production rate vanishes if and only if the distribution function is a Maxwellian.

1 Background

The collision operator for collisions of a given species with itself is

$$C = \gamma \nabla_v \cdot \int d^3v' \, \mathbf{U} \cdot \left(f' \nabla_v f - f \nabla_{v'} f' \right) \tag{1}$$

where

$$\gamma = \frac{q^4 \ln \Lambda}{4\pi\epsilon_0 m^2}, \tag{2}$$

$$\mathbf{U} = \frac{1}{u^3}\left(u^2 \mathbf{I} - \boldsymbol{u}\boldsymbol{u} \right), \tag{3}$$

$$\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}', \tag{4}$$

and $u = |\boldsymbol{u}|$. The distribution function evaluated at $\boldsymbol{v}$ is denoted f, as shorthand for $f(\boldsymbol{v})$. Similarly, f' denotes $f(\boldsymbol{v}')$, the distribution function evaluated at $\boldsymbol{v}'$, and $\nabla_{v'}$ denotes the velocity-space gradient with respect to $\boldsymbol{v}'$.

The entropy associated with a distribution function is

$$S = -\int d^3v \; f \ln f. \tag{5}$$

Evaluating the entropy production rate $\dot{S} = dS/dt$ by taking the time derivative of (5) and plugging in the Fokker-Planck equation for $\partial f / \partial t$, all the terms from the collisionless part of the Fokker-Planck equation end up vanishing, as does a term $-\int d^3v \; C$ which vanishes due to particle conservation, and the only remaining term is

$$\dot{S} = -\int d^3v \; (\ln f) C. \tag{6}$$

This equation therefore gives the rate of entropy production.

2 Proof of Theorem 1

Let us now prove that $\dot{S} \geq 0$. Plugging (1) into (6),

$$\dot{S} = -\gamma \int d^3v \, (\ln f) \nabla_v \cdot \int d^3v' \mathbf{U} \cdot \left(f' \nabla_v f - f \nabla_{v'} f' \right) . \tag{7}$$

Integrating by parts in $\boldsymbol{v}$,

$$\dot{S} = \gamma \int d^3v \int d^3v' \ \left(\nabla_v \ln f \right) \cdot \mathbf{U} \cdot \left(f' \nabla_v f - f \nabla_{v'} f' \right) . \tag{8}$$

We then use

$$\nabla_v f = f \nabla_v \ln f, \tag{9}$$

$$\nabla_{v'} f' = f' \nabla_{v'} \ln f', \tag{10}$$

to obtain

$$\dot{S} = \gamma \int d^3v \int d^3v' \ f f' \left(\nabla_v \ln f \right) \cdot \mathbf{U} \cdot \left(\nabla_v \ln f - \nabla_{v'} \ln f' \right) . \tag{11}$$

We are free to interchange the dummy integration variables $\boldsymbol{v}$ and $\boldsymbol{v}'$, giving

$$\dot{S} = -\gamma \int d^3v \int d^3v' \ f f' \left(\nabla_{v'} \ln f' \right) \cdot \mathbf{U} \cdot \left(\nabla_v \ln f - \nabla_{v'} \ln f' \right) . \tag{12}$$

We then take an average of (11) and (12), yielding

$$\dot{S} = \frac{\gamma}{2} \int d^3v \int d^3v' \ f f' \left(\nabla_v \ln f - \nabla_{v'} \ln f' \right) \cdot \mathbf{U} \cdot \left(\nabla_v \ln f - \nabla_{v'} \ln f' \right) . \tag{13}$$

Defining

$$\boldsymbol{H}(\boldsymbol{v}, \boldsymbol{v}') = \nabla_v \ln f - \nabla_{v'} \ln f', \tag{14}$$

we see that (13) contains the quantity

$$\boldsymbol{H} \cdot \mathbf{U} \cdot \boldsymbol{H} = \frac{1}{u^3} \left(u^2 H^2 - [\boldsymbol{u} \cdot \boldsymbol{H}]^2 \right) , \tag{15}$$

where $H = |\boldsymbol{H}|$. This quantity in (15) must be nonnegative due to the triangle inequality in $\mathbb{R}^3$. In other words, we know

$$\boldsymbol{u} \cdot \boldsymbol{H} = uH \cos\theta \tag{16}$$

where θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{H}$, and so $[\boldsymbol{u} \cdot \boldsymbol{H}]^2 \leq u^2 H^2$.

The remaining quantities in (13) (γ, f, and f') are all nonnegative as well. Hence, (13) shows that $\dot{S}$ is an integral of a nonnegative integrand, and hence the integral is nonnegative. This proves theorem 1.

3 Proof of Theorem 2

Let us next prove that $\dot{S} = 0$ if and only if $\boldsymbol{f}$ is a Maxwellian distribution function. There are two parts to this statement, 'if' and 'only if', and we will prove the 'if' statement first.

A distribution function $f(\boldsymbol{v})$ is defined to be a Maxwellian if it depends on $\boldsymbol{v}$ only through the form

$$f(\boldsymbol{v}) = n \left[\frac{m}{2\pi T} \right]^{3/2} \exp \left(-\frac{m}{2T} [\boldsymbol{v} - \boldsymbol{w}]^2 \right) \tag{17}$$

where n is the density, T is the temperature, and $\boldsymbol{w}$ is the mean flow velocity. Then

$$\nabla_v \ln f = -\frac{m}{T}[\boldsymbol{v} - \boldsymbol{w}], \tag{18}$$

and so

$$\boldsymbol{H} = \nabla_v \ln f - \nabla_{v'} \ln f' = \frac{m}{T}[\boldsymbol{v}' - \boldsymbol{v}] = -\frac{m}{T}\boldsymbol{u}. \tag{19}$$

Then

$$\mathbf{U} \cdot \boldsymbol{H} = -\frac{m}{Tu^3}\left[u^2 \mathbf{I} - \boldsymbol{u}\boldsymbol{u}\right] \cdot \boldsymbol{u} = 0. \tag{20}$$

Hence, (13) vanishes. This proves the 'if' part of Theorem 2.

Let us now prove the 'only if' part of theorem 2. That is, if we set (13) to zero, we want to derive that f is Maxwellian. Since γ, f, f', and $\boldsymbol{H} \cdot \mathbf{U} \cdot \boldsymbol{H}$ are all nonnegative, the only way that (13) can vanish is if $\boldsymbol{H} \cdot \mathbf{U} \cdot \boldsymbol{H} = 0$ for all $\boldsymbol{v}$ and $\boldsymbol{v}'$ (where f and f' are nonzero.) Using (15) and (16), we see that $\boldsymbol{H} \cdot \mathbf{U} \cdot \boldsymbol{H} = 0$ implies $\theta = 0$ or $\theta = \pi$, i.e.

$$\boldsymbol{u} || \boldsymbol{H} \quad \text{for all } \boldsymbol{v},\ \boldsymbol{v}'. \tag{21}$$

An equivalent statement is that there exists some scalar $\lambda = \lambda(\boldsymbol{v}, \boldsymbol{v}')$ such that

$$\lambda \boldsymbol{u} = \boldsymbol{H} \quad \text{for all } \boldsymbol{v},\ \boldsymbol{v}'. \tag{22}$$

We can prove that λ must in fact be independent of $\boldsymbol{v}$ and $\boldsymbol{v}'$, but this is a bit tricky to prove. Here is one approach. The z component of (22) is

$$(v_z - v_z')\lambda = \frac{\partial \ln f}{\partial v_z} - \frac{\partial \ln f'}{\partial v_z'}. \tag{23}$$

Applying $\partial/\partial v_x$, we get

$$(v_z - v_z')\frac{\partial \lambda}{\partial v_x} = \frac{\partial^2 \ln f}{\partial v_x \partial v_z}, \tag{24}$$

and so

$$\frac{\partial \lambda}{\partial v_x} = \frac{1}{v_z - v_z'}\frac{\partial^2 \ln f}{\partial v_x \partial v_z}, \tag{25}$$

where we have used the fact that f' is independent of v_x. We can repeat the last three steps starting with the y component of (22) instead of the z component, in which case we get

$$\frac{\partial \lambda}{\partial v_x} = \frac{1}{v_y - v_y'}\frac{\partial^2 \ln f}{\partial v_x \partial v_y}. \tag{26}$$

We can equate (25)-(26), giving

$$\frac{1}{v_y - v_y'}\frac{\partial^2 \ln f}{\partial v_x \partial v_y} = \frac{1}{v_z - v_z'}\frac{\partial^2 \ln f}{\partial v_x \partial v_z}. \tag{27}$$

Notice the left-hand side depends on v_y', while the right-hand side does not. The only way these dependencies can be consistent is if the v_y' dependence on the left is multiplied by 0, i.e.

$$\frac{\partial^2 \ln f}{\partial v_x \partial v_y} = 0. \tag{28}$$

Plugging this result into (26), we find $\partial\lambda/\partial v_x = 0$.

There was nothing special about the x component in the derivation of $\partial\lambda/\partial v_x = 0$, so we can repeat the steps choosing different velocity components to prove $\partial\lambda/\partial v_y = 0$ and $\partial\lambda/\partial v_z = 0$. We can also repeat the steps with the roles of $\boldsymbol{v}$ and $\boldsymbol{v}'$ reversed to prove that $\partial\lambda/\partial v'_x = \partial\lambda/\partial v'_y = \partial\lambda/\partial v'_z = 0$. Hence, λ is in fact a constant (independent of $\boldsymbol{v}$ and $\boldsymbol{v}'$.)

We next write (23) as

$$\frac{\partial \ln f}{\partial v_z} - \lambda v_z = \frac{\partial \ln f'}{\partial v'_z} - \lambda v'_z. \tag{29}$$

The left-hand side depends only on $\boldsymbol{v}$ and is independent of $\boldsymbol{v}'$. Conversely, the right-hand side depends only on $\boldsymbol{v}'$ and is independent of $\boldsymbol{v}$. The only way these dependencies can be reconciled is if the left and right sides independently equal a constant. Let us choose to denote this constant by $-\lambda w_z$:

$$\frac{\partial \ln f}{\partial v_z} - \lambda v_z = -\lambda w_z. \tag{30}$$

Integrating in v_z,

$$\ln f = \frac{\lambda}{2}[v_z - w_z]^2 + F_{xy}(v_x, v_y), \tag{31}$$

where $F_{xy}(v_x, v_y)$ is an integration 'constant' (meaning it is constant with respect to the integration variable v_z, but it is free to depend on the other quantities v_x and v_y.) If we repeated the steps from (29) but choosing to start with the x or y components instead of the z component, we would get the following equations analogous to (31):

$$\ln f = \frac{\lambda}{2}[v_x - w_x]^2 + F_{yz}(v_y, v_z), \tag{32}$$

$$= \frac{\lambda}{2}[v_y - w_y]^2 + F_{xz}(v_x, v_z), \tag{33}$$

where w_x and w_y are constants, and F_{yz} and F_{xz} are some undetermined functions. A general way to reconcile the dependencies of $\ln f$ on v_x, v_y, and v_z given in (31)-(33) is

$$\ln f = \frac{\lambda}{2}[\boldsymbol{v} - \boldsymbol{w}]^2 + k, \tag{34}$$

where k is independent of $\boldsymbol{v}$. Applying $\exp(\ldots)$, we find that f can depend on $\boldsymbol{v}$ only through the form of a Maxwellian, (17). We can identify the temperature as $T = -m/\lambda$. This proves theorem 2.