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More accurate ion flow coefficients in the Pfirsch-Schluter regime

In these notes we repeat and slightly extend the calculation of Ref. [1], computing the ion neoclassical flow in the Pfirsch-Schluter (short-mean-free-path) collisionality regime. The calculation relies on the solution of several generalized Spitzer problems. In the original work [1], the Spitzer problems were solved using a 2-Laguerre-polynomial approximation. Here we derive the flow in terms of general integrals of Spitzer functions, so the coefficients may be evaluated as accurately as one wishes. The main reason for more accurate evaluation of these coefficients is for benchmarking of neoclassical codes in the limit of high collisionality.

The final result of Ref. [1] is that the parallel ion flow is

$$nV_{\|} = -\frac{IcnT}{ZeB}\left(\frac{p'}{p}+\frac{Ze\Phi'}{T}-k\frac{B^2}{\langle B^2\rangle}\frac{T'}{T}\right) \tag{1}$$

where primes denote $d/d\psi$,

$$k = k_0 + k_1\frac{\langle B^2\rangle\left\langle\left(\nabla_{\|}\ln B\right)^2\right\rangle}{\left\langle\left(\nabla_{\|}B\right)^2\right\rangle}, \tag{2}$$

and the two dimensionless flow coefficients are

$$\begin{aligned} k_0 &= -1.8, \\ k_1 &= -0.27. \end{aligned} \tag{3}$$

In eq (22) of Ref. [2], more accurate values of the coefficients are given as

$$\begin{aligned} k_0 &= -1.77, \\ k_1 &= -0.05. \end{aligned} \tag{4}$$

The main result of the present notes is that as the accuracy of the υ discretization is increased, (i.e. as more accurate Spitzer functions are used), the flow coefficients converge to

$$\boxed{\begin{aligned} k_0 &= -1.97213, \\ k_1 &= -0.09361. \end{aligned}} \tag{5}$$

To a good approximation, the ratio of flux surface averages in (2) is ≈ 1, and so $k \approx k_0 + k_1 = -2.06574$.

The calculation here (as in [1]) neglects ion-electron collisions. Note that if ion-electron collisions were included, the ion flow would change by $\sim\sqrt{m_e/m_i}$.

Preliminary steps

We wish to solve the drift-kinetic equation

$$\upsilon_{\|}\nabla_{\|}f_1 - C\{f_1\} = -\mathbf{v}_d\cdot\nabla\psi\frac{\partial f_0}{\partial\psi} \tag{6}$$

for high collisionality: $\nu_{ii}qR/\upsilon_i \gg 1$ where $\upsilon_i = \sqrt{2T/m}$ is the thermal speed and

$$\nu_{ii} = \frac{4\sqrt{2\pi} n e^4 \ln\Lambda}{3 m^{1/2} T^{3/2}} \tag{7}$$

is the ion-ion collision frequency. The gradient in (6) is performed at fixed $\mu = \upsilon^2 \left(1-\xi^2\right)/\left(2B\right)$ where $\xi = \upsilon_\| / \upsilon$. Also, $\mathbf{v}_d$ denotes the magnetic drift, and

$$f_0 = \frac{n}{\pi^{3/2} \upsilon_i^3} e^{-x^2} \tag{8}$$

is the leading-order Maxwellian, with $x = \upsilon / \upsilon_i$. We take C to be the ion-ion collision operator, neglecting ion-electron collisions. It will be convenient to expand the right-hand side of (6) as

$$\upsilon_\| \nabla_\| f_1 - C\{f_1\} = -\frac{mcI\upsilon^2\left(1+\xi^2\right)}{4Ze}\left(\mathbf{B}\cdot\nabla\frac{1}{B^2}\right) f_0 \left(\frac{p'}{p} + \frac{Ze\Phi'}{T} + \left[x^2 - \frac{5}{2}\right]\frac{T'}{T}\right). \tag{9}$$

We will need to annihilate the largest term, the collision operator, which is done by applying $\int d^3\upsilon(\ldots)$, $\int d^3\upsilon\, \upsilon_\|(\ldots)$, and $\int d^3\upsilon\, \left(m\upsilon^2/2 - 5T/2\right)(\ldots)$, yielding moment equations. These three moment equations are

$$\nabla\cdot\int d^3\upsilon\; f_1 \upsilon_\| \mathbf{b} = -\mathbf{B}\cdot\nabla\left[\frac{1}{B^2}\frac{IcnT}{Ze}\left(\frac{p'}{p} + \frac{Ze\Phi'}{T}\right)\right], \tag{10}$$

$$\nabla_\|\left(\int d^3\upsilon\; \upsilon_\|^2 f_1\right) + \left[\int d^3\upsilon\; f_1 \upsilon^2\left(1-3\xi^2\right)\right]\frac{1}{2B}\nabla_\| B = 0, \tag{11}$$

and

$$\nabla\cdot\int d^3\upsilon\; f_1 \upsilon_\| \left(\frac{\upsilon^2}{2} - \frac{5T}{2m}\right)\mathbf{b} = -\frac{5}{2} n \frac{T^2}{m^2}\frac{mcI}{Ze}\left(\mathbf{B}\cdot\nabla\frac{1}{B^2}\right)\frac{T'}{T}. \tag{12}$$

It will be frequently useful in the following analysis to apply $\left\langle \mathbf{B}\cdot(\ldots)\right\rangle$ to (11) to obtain

$$\left\langle \left[\int d^3\upsilon\; f_1 \upsilon^2 \left(1-3\xi^2\right)\right]\nabla_\| B\right\rangle = 0. \tag{13}$$

Here $\langle\ \rangle$ denotes a flux-surface average.

We expand $f_1 = f_1^{(-1)} + f_1^{(0)} + f_1^{(1)} + \ldots$ where $f_1^{(j)} \approx \nu_{ii}^{-j}$. We begin the series with $j=-1$ since it turns out that f_1 will have a term $\propto \nu_{ii}$.

Order ν_{ii}^2 kinetic equation:

The leading order term in (6) is

$$0 = C\left\{f_1^{-1}\right\}. \tag{14}$$

Therefore

$$f_1^{-1} = \left[\frac{p^{(-1)}(\theta)}{p} + \frac{m}{T}\upsilon_\| V^{(-1)}(\theta) + \left(x^2 - \frac{5}{2}\right)\frac{T^{(-1)}(\theta)}{T}\right] f_0 \tag{15}$$

for some functions $p^{(-1)}$, $V^{(-1)}$, and $T^{(-1)}$.

Order ν_{ii}^1 constraints:

Constraint (10) gives

$$\mathbf{B}\cdot\nabla\left(\frac{V^{(-1)}}{B}\right)=0 \tag{16}$$

so

$$V^{(-1)}=BA_1(\psi) \tag{17}$$

for some flux function A_1. Constraint (12) tells us nothing at this order. Constraint (11) tells us

$$\nabla_\parallel p^{(-1)}=0. \tag{18}$$

We may as well have kept all the flux-surface-averaged pressure in f_0, so $p^{(-1)}=0$.

Order ν_{ii}^1 kinetic equation:

The terms in (6) $\propto \nu$ are

$$\upsilon_\parallel\nabla_\parallel f_1^{(-1)}-C\left\{f_1^{(0)}\right\}=0. \tag{19}$$

Plugging in (15),

$$f_0x^2\left(3\xi^2-1\right)A_1\nabla_\parallel B+f_0\left(x^2-\frac{5}{2}\right)x\xi\frac{\upsilon_i}{T}\nabla_\parallel T^{(-1)}=C\left\{f_1^{(0)}\right\}. \tag{20}$$

Notice that the first few Legendre polynomials are

$$\begin{aligned}P_0(\xi)&=1,\\ P_1(\xi)&=\xi,\\ P_2(\xi)&=\frac{1}{2}\left(3\xi^2-1\right).\end{aligned} \tag{21}$$

Since Legendre polynomials are eigenfunctions of C, the solution of (19) must be

$$\begin{aligned}f_1^{(0)}=&\left[\frac{p^{(0)}(\theta)}{p}+\frac{m}{T}\upsilon_\parallel V^{(0)}(\theta)+\left(x^2-\frac{5}{2}\right)\frac{T^{(0)}(\theta)}{T}\right]f_0\\ &+F_1(x)\,f_0\xi\frac{\upsilon_i}{\nu_{ii}T}\nabla_\parallel T^{(-1)}+F_2(x)\,f_0\left(3\xi^2-1\right)\frac{A_1\nabla_\parallel B}{\nu_{ii}}\end{aligned} \tag{22}$$

where $p^{(0)}$, $V^{(0)}$, and $T^{(0)}$ are new functions associated with the homogeneous solution, and $F_1(x)$ and $F_2(x)$ are defined by the Spitzer problems

$$\frac{1}{\nu_{ii}}C\left\{F_1(x)\,\xi f_0\right\}=\left(x^2-\frac{5}{2}\right)x\xi f_0 \tag{23}$$

and

$$\frac{1}{\nu_{ii}}C\left\{F_2(x)\left(3\xi^2-1\right)f_0\right\}=x^2\left(3\xi^2-1\right)f_0. \tag{24}$$

Approximate solution of first Spitzer problem:

Although we ultimate want to compute the Spitzer functions to arbitrarily high accuracy, to make contact with Ref. [1], here we repeat the 2-polynomial approximation used there.

It will turn out that we don't need the solution of (24) (for now), but we will need the solution of (23). Notice that we can always add a constant to F_1, since $C\{x\xi f_0\}=0$. We consider an expansion in Laguerre polynomials of order 3/2, keeping the first 2 terms after the constant:

$$\begin{aligned} F_1(x) &= x\sum_{k=1} a_k L_k^{(3/2)}\left(x^2\right) \\ &= a_1\left(\frac{5}{2}-x^2\right)x + a_2\frac{1}{8}\left(35-28x^2+4x^4\right)x. \end{aligned} \tag{25}$$

Multiplying (23) by another Laguerre polynomial and integrating over velocity space,

$$\int d^3\upsilon x\xi L_j^{(3/2)}(x)\, f_0\left(x^2-\frac{5}{2}\right)x\xi = \frac{1}{\nu_{ii}}\int d^3\upsilon x\xi L_j^{(3/2)}(x) C\left\{ f_0\xi x\sum_k a_k L_k^{(3/2)}\left(x^2\right)\right\}. \tag{26}$$

From (B6) of Ref. [3], the right-hand side of (26) is

$$-\frac{n}{\sqrt{2}}\begin{pmatrix} 1 & 3/4 \\ 3/4 & 45/16 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{27}$$

The left-hand side of (26) is

$$\begin{aligned} \int d^3\upsilon x\xi L_j^{(3/2)}(x)\, f_0\left(x^2-\frac{5}{2}\right)x\xi &= \frac{4n}{3\pi^{1/2}}\int_0^\infty dx\, x^4 L_j^{(3/2)}(x) e^{-x^2}\left(x^2-\frac{5}{2}\right) \\ &= -\frac{5}{4}n\delta_{j,1}. \end{aligned} \tag{28}$$

Thus, we have a linear system

$$\frac{5\sqrt{2}}{4}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 3/4 \\ 3/4 & 45/16 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{29}$$

The solution is

$$\begin{aligned} a_1 &= \frac{25}{16}\sqrt{2} \\ a_2 &= -\frac{5}{12}\sqrt{2} \end{aligned} \tag{30}$$

so

$$\begin{aligned} F_1(x) &= \left[\frac{25}{16}\left(\frac{5}{2}-x^2\right) - \frac{5}{12}\frac{1}{8}\left(35-28x^2+4x^4\right)\right]x\sqrt{2} \\ &= \left[\frac{25}{16}\frac{5}{2} - \frac{5}{12}\frac{35}{8} - \frac{5}{48}\left(x^2+2x^4\right)\right]x\sqrt{2} \end{aligned} \tag{31}$$

As noted above, we can shift F_1 by a constant, so we may simplify it to

$$F_1(x) = -\frac{5}{48}\left(x^2+2x^4\right)x\sqrt{2} \tag{32}$$

making (22) into

$$f_1^{(0)} = -\frac{5}{48}\left(x^2+2x^4\right)x\sqrt{2}\, f_0\xi\frac{\upsilon_i}{\nu_{ii}T}\nabla_\parallel T^{(-1)} + \ldots \tag{33}$$

Notice Ref. [1] uses the collision time

$$\tau = \frac{3m^2 \upsilon_i^3}{8\sqrt{2\pi} n e^4 \ln\Lambda} = \frac{3m^{1/2} T^{3/2}}{4\sqrt{\pi} n e^4 \ln\Lambda} = \frac{\sqrt{2}}{\nu_{ii}} \tag{34}$$

so (33) becomes

$$f_1^{(0)} = -\frac{5}{48}\left(x^2 + 2x^4\right) x f_0 \xi \frac{\tau \upsilon_i}{T} \nabla_\| T^{(-1)} + \dots \tag{35}$$

in agreement with eq (51)-(52) in Ref. [1].

Order ν_{ii}^0 constraints:

Plugging (22) into (13) gives

$$A_1 \left[\int d^3\upsilon \; S_2(\upsilon) \upsilon^2 \left(1 - 3\xi^2\right)^2 \right] \left\langle \left(\nabla_\| B\right)^2 \right\rangle = 0. \tag{36}$$

In general, the υ integral and $\left\langle \left(\nabla_\| B\right)^2 \right\rangle$ do not vanish, so A_1 must be 0. Thus,

$$f_1^{-1} = \left(x^2 - \frac{5}{2}\right) \frac{T^{(-1)}(\theta)}{T} f_0 . \tag{37}$$

Plugging (22) into (11) then gives $\nabla_\| p^{(0)} = 0$, so again it is no loss in generality to take $p^{(0)} = 0$, as we can absorb it into f_0. Thus, we have

$$f_1^{(0)} = \frac{m}{T} \upsilon_\| V^{(0)} f_0 + F_1(x) \, f_0 \xi \frac{\upsilon_i}{\nu_{ii} T} \nabla_\| T^{(-1)} + \left(x^2 - \frac{5}{2}\right) \frac{T^{(0)}}{T} f_0 . \tag{38}$$

We next plug this result into (12), finding

$$\frac{1}{3} \nabla \cdot \left[\int d^3\upsilon \; F_1(x) \, f_0 \frac{\upsilon_i^2}{\nu_{ii} T} \left(\nabla_\| T^{(-1)}\right) x \left(\frac{\upsilon^2}{2} - \frac{5T}{2m}\right) \mathbf{b} \right] = -\frac{5}{2} n \frac{T^2}{m^2} \frac{mcI}{Ze} \left(\mathbf{B} \cdot \nabla \frac{1}{B^2}\right) \frac{T'}{T} \tag{39}$$

This result is basically just a statement of the incompressibility of the heat flux: $0 = \nabla \cdot \left(q_\| \mathbf{b} + \mathbf{q}_{diamag}\right)$ for the classical $q_\| \propto \nabla_\| T$ where $\mathbf{q}_{diamag}$ is the diamagnetic heat flux. Defining a dimensionless coefficient

$$X_1 = \int_0^\infty dx \; F_1(x) e^{-x^2} x^3 \left(x^2 - \frac{5}{2}\right) \tag{40}$$

then

$$\mathbf{B} \cdot \nabla \left[\frac{\nabla_\| T^{(-1)}}{B} + \frac{15\sqrt{\pi}}{16 X_1} \frac{\nu_{ii} T mcI}{Ze} \frac{1}{B^2} \frac{T'}{T} \right] = 0. \tag{41}$$

Consequently

$$\nabla_\| T^{(-1)} + \frac{15\sqrt{\pi}}{16 X_1} \frac{\nu_{ii} T mcI}{Ze} \frac{1}{B} \frac{T'}{T} = B A_2(\psi) \tag{42}$$

for some $A_2(\psi)$. Applying $\langle B(\dots) \rangle$,

$$\frac{B}{\langle B^2 \rangle} \frac{15\sqrt{\pi}}{16 X_1} \frac{\nu_{ii} T mcI}{Ze} \frac{T'}{T} = B A_2(\psi) \tag{43}$$

so

$$\nabla_{\|} T^{(-1)} = -\frac{15\sqrt{\pi}}{16X_1}\frac{\nu_{ii} TmcI}{Ze}\frac{1}{B}\left(1-\frac{B^2}{\langle B^2\rangle}\right)\frac{T'}{T}. \tag{44}$$

For the 2-polynomial approximation (32), then

$$X_1 = -\frac{5}{48}\sqrt{2}\int_0^\infty dx e^{-x^2} x^4\left(x^2-\frac{5}{2}\right)\left(x^2+2x^4\right) = -2.60\sqrt{2} \tag{45}$$

so (44) becomes

$$\nabla_{\|} T^{(-1)} = 0.68\frac{\nu_{ii} TmcI}{\sqrt{2}ZeB}\left(1-\frac{B^2}{\langle B^2\rangle}\right)\frac{T'}{T} \tag{46}$$

in agreement with eq (53) in [1].

Next, we observe from (38) that the actual parallel flow is not $V^{(0)}$ but rather

$$\begin{aligned} nV_{\|} &= \int d^3\upsilon \upsilon_{\|} f_1 = nV^{(0)} + \frac{4X_2 n}{3\sqrt{\pi}}\frac{\upsilon_i^2}{\nu_{ii} T}\nabla_{\|} T^{(-1)} \\ &= nV^{(0)} - \frac{5X_2}{2X_1}\frac{TcnI}{Ze}\frac{1}{B}\left(1-\frac{B^2}{\langle B^2\rangle}\right)\frac{T'}{T}. \end{aligned} \tag{47}$$

where

$$X_2 = \int_0^\infty dx\, F_1(x)\, x^3 e^{-x^2} \tag{48}$$

is another dimensionless constant.

Plugging (47) into (10), we find

$$\mathbf{B}\cdot\nabla\left[\frac{nV^{(0)}}{B} + \frac{n\upsilon_i^2}{B\nu_{ii} T}\nabla_{\|} T^{(-1)}\frac{4}{3\sqrt{\pi}}X_2 + \frac{1}{B^2}\frac{IcnT}{Ze}\left(\frac{p'}{p}+\frac{Ze\Phi'}{T}\right)\right] = 0. \tag{49}$$

Then

$$nV^{(0)} = BA_3(\psi) + \frac{5X_2}{2X_1}\frac{TcnI}{Ze}\frac{1}{B}\left(1-\frac{B^2}{\langle B^2\rangle}\right)\frac{T'}{T} - \frac{1}{B}\frac{IcnT}{Ze}\left(\frac{p'}{p}+\frac{Ze\Phi'}{T}\right) \tag{50}$$

for some flux function $A_3(\psi)$. Notice the total parallel flow (47) is

$$nV_{\|} = BA_3(\psi) - \frac{1}{B}\frac{IcnT}{Ze}\left(\frac{p'}{p}+\frac{Ze\Phi'}{T}\right). \tag{51}$$

Comparing this expression to (1), we can define the parallel flow coefficient k in terms of A_3:

$$k = \frac{A_3\langle B^2\rangle}{\dfrac{IcnT}{Ze}\dfrac{T'}{T}}. \tag{52}$$

For the 2-polynomial approximation (32),

$$X_2 = -\frac{25}{16}\sqrt{\frac{\pi}{2}} \tag{53}$$

so the numerical coefficient in (50) is

$$\frac{5X_2}{2X_1} = 1.333 \tag{54}$$

in agreement with eq (54) in Ref. [1].

Order ν_{ii}^0 kinetic equation:

The terms in (6) $\propto \nu^0$ are

$$\upsilon_\| \nabla_\| f_1^0 - C\left\{f_1^1\right\} = -\frac{mcI\upsilon^2\left(1+\xi^2\right)}{4Ze}\left(\mathbf{B}\cdot\nabla\frac{1}{B^2}\right) f_0\left(\frac{p'}{p}+\frac{Ze\Phi'}{T}+\left[x^2-\frac{5}{2}\right]\frac{T'}{T}\right). \tag{55}$$

Expanding the first term,

$$C\left\{f_1^1\right\} = \frac{mcI\upsilon^2\left(1+\xi^2\right)}{4Ze}\left(\mathbf{B}\cdot\nabla\frac{1}{B^2}\right) f_0\left(\frac{p'}{p}+\frac{Ze\Phi'}{T}+\left[x^2-\frac{5}{2}\right]\frac{T'}{T}\right) + \left[\begin{array}{l} \frac{m}{T}\left(\upsilon_\|\nabla_\|\upsilon_\|\right)V^{(0)} f_0 + f_0\frac{m}{T}\upsilon_\|^2\nabla_\| V^{(0)} + \frac{Y_0^1(x)}{x} f_0\left(\upsilon_\|\nabla_\|\upsilon_\|\right)\frac{1}{\nu_{ii}T}\nabla_\| T^{(-1)} \\ +\frac{F_1(x)}{x} f_0\frac{1}{\nu_{ii}T}\upsilon_\|^2\nabla_\|\nabla_\| T^{(-1)} + \left(x^2-\frac{5}{2}\right)\frac{\upsilon_\|\nabla_\| T^{(0)}}{T} f_0 \end{array}\right]. \tag{56}$$

To evaluate several of the terms, we can recall

$$\upsilon_\|\nabla_\|\upsilon_\| = -\frac{\upsilon^2\left(1-\xi^2\right)}{2B}\left(\nabla_\| B\right). \tag{57}$$

It is apparent that the right-hand side of (56) has $L_j(\xi)$ terms for j=0,1,2. The solution is

$$f_1^1 = \frac{1}{\nu_{ii}}\left[Y_0(x)L_0(\xi)+Y_1(x)L_1(\xi)+Y_2(x)L_2(\xi)\right] f_0 . \tag{58}$$

It turns out that only the $j=2$ term is needed to compute the flow. Thus,

$$C\left\{f_1^1\right\} = \frac{2}{3}\underbrace{\frac{1}{2}\left(3\xi^2-1\right)}_{L_2(\xi)}\left\{\begin{array}{l} \frac{mcI\upsilon^2}{4Ze}\left(\mathbf{B}\cdot\nabla\frac{1}{B^2}\right) f_0\left(\frac{p'}{p}+\frac{Ze\Phi'}{T}+\left[x^2-\frac{5}{2}\right]\frac{T'}{T}\right) \\ +\left[\begin{array}{l} +\frac{m}{T}\frac{\upsilon^2}{2B}\left(\nabla_\| B\right)V^{(0)} f_0 + f_0\frac{m}{T}\upsilon_i^2 x^2\nabla_\| V^{(0)} + Y_0^1(x)\, x f_0\frac{\upsilon_i^2}{2B}\left(\nabla_\| B\right)\frac{1}{\nu_{ii}T}\nabla_\| T^{(-1)} \\ +F_1(x) f_0 x\frac{\upsilon_i^2}{\nu_{ii}T}\nabla_\|\nabla_\| T^{(-1)} \end{array}\right] \end{array}\right\} + L_{j\neq 2} . \tag{59}$$

The terms in the right-hand side of (59) have 3 basic types of x-dependence, so there are 3 basic Spitzer problems we must solve:

$$\frac{1}{\nu_{ii}} C\left\{f_0 F_2(x) L_2(\xi)\right\} = L_2(\xi) f_0 x^2 , \tag{60}$$

$$\frac{1}{\nu_{ii}} C\left\{f_0 F_4(x) L_2(\xi)\right\} = L_2(\xi) f_0 x^4 , \tag{61}$$

$$\frac{1}{\nu_{ii}} C\left\{f_0 F_F(x) L_2(\xi)\right\} = L_2(\xi) f_0 x F_1(x), \tag{62}$$

for the functions F_2, F_4, and F_F. The solution to (58)-(59) then has the form

$$Y_2(x) = d_2 F_2(x) + d_4 F_4(x) + d_F F_F(x) \tag{63}$$

where

$$d_2 = \frac{2}{3}\left\{ \frac{mcI\upsilon_i^2}{4Ze}\left(\mathbf{B}\cdot\nabla\frac{1}{B^2}\right)\left(\frac{p'}{p} + \frac{Ze\Phi'}{T} - \frac{5}{2}\frac{T'}{T}\right) + \frac{1}{B}\left(\nabla_\parallel B\right)V^{(0)} + 2\nabla_\parallel V^{(0)}\right\}, \tag{64}$$

$$d_4 = -\frac{2}{3}\frac{TcI}{ZeB^2}\left(\nabla_\parallel B\right)\frac{T'}{T}, \tag{65}$$

$$\begin{aligned} d_F &= \frac{2}{3}\frac{\upsilon_i^2}{\nu_{ii}T}\left[\frac{1}{2B}\left(\nabla_\parallel B\right)\nabla_\parallel T^{(-1)} + \nabla_\parallel \nabla_\parallel T^{(-1)}\right] \\ &= \frac{2}{3}\left(\nabla_\parallel B\right)\left[\frac{1}{B^2} + \frac{3}{\langle B^2\rangle}\right]\frac{15\sqrt{\pi}}{16X_1}\frac{TcI}{Ze}\frac{T'}{T}. \end{aligned} \tag{66}$$

With a bit of manipulation, it can be seen that the p' and Φ' terms in (64) cancel, leaving

$$d_2 = 2\left(\nabla_\parallel B\right)\frac{A_3}{n} + \frac{5}{3}\left(\nabla_\parallel B\right)\left[\frac{1}{B^2} + \frac{X_2}{X_1}\left(-\frac{1}{B^2} - \frac{3}{\langle B^2\rangle}\right)\right]\frac{TcI}{Ze}\frac{T'}{T} \tag{67}$$

Order ν_{ii}^{-1} constraint:

At this order, the only constraint we need to consider is (13). Due to the orthogonality of the Legendre polynomials, only the $L_2(\xi)$ component of f_1^1 matters in this constraint, so we find

$$\left\langle \left[\int_0^\infty dx\; x^4 e^{-x^2}\left(d_2F_2 + d_4F_4 + d_F F_F\right)\right]\left(\nabla_\parallel B\right)\right\rangle = 0. \tag{68}$$

We define the dimensionless coefficients

$$\begin{aligned} X_3 &= \int_0^\infty dx\; x^4 e^{-x^2} F_2(x) \\ X_4 &= \int_0^\infty dx\; x^4 e^{-x^2} F_4(x) \\ X_5 &= \int_0^\infty dx\; x^4 e^{-x^2} F_F(x) \end{aligned} \tag{69}$$

so

$$\left\langle \left(d_2X_3 + d_4X_4 + d_F X_5\right)\left(\nabla_\parallel B\right)\right\rangle = 0. \tag{70}$$

$$\left\langle \left(A_3 + \left[\begin{aligned} &\left(\frac{5}{6} - \frac{5X_2}{6X_1} - \frac{X_4}{3X_3} + \frac{X_5}{X_3}\frac{5\sqrt{\pi}}{16X_1}\right)\frac{1}{B^2} \\ &+\left(-\frac{5X_2}{2X_1} + \frac{X_5}{X_3}\frac{15\sqrt{\pi}}{16X_1}\right)\frac{1}{\langle B^2\rangle} \end{aligned} \right]\frac{TcnI}{Ze}\frac{T'}{T}\right)\left(\nabla_\parallel B\right)^2 \right\rangle = 0. \tag{71}$$

This constraint has the form

$$\left\langle \left[A_3 + \frac{Q_1}{B^2} + \frac{Q_2}{\langle B^2 \rangle} \right] \left(\nabla_{\parallel} B \right)^2 \right\rangle = 0 \tag{72}$$

where

$$Q_1 = \left(\frac{5}{6} - \frac{5X_2}{6X_1} - \frac{X_4}{3X_3} + \frac{X_5}{X_3} \frac{5\sqrt{\pi}}{16X_1} \right) \frac{TcnI}{Ze} \frac{T'}{T} \tag{73}$$

and

$$Q_2 = \left(-\frac{5X_2}{2X_1} + \frac{X_5}{X_3} \frac{15\sqrt{\pi}}{16X_1} \right) \frac{TcnI}{Ze} \frac{T'}{T}. \tag{74}$$

The solution is

$$A_3 = -Q_1 \frac{\left\langle \left(\nabla_{\parallel} \ln B \right)^2 \right\rangle}{\left\langle \left(\nabla_{\parallel} B \right)^2 \right\rangle} - Q_2 \frac{1}{\langle B^2 \rangle}. \tag{75}$$

Thus, from (52),

$$k = -\frac{1}{\frac{IcnT}{Ze} \frac{T'}{T}} \left[Q_1 \frac{\langle B^2 \rangle \left\langle \left(\nabla_{\parallel} \ln B \right)^2 \right\rangle}{\left\langle \left(\nabla_{\parallel} B \right)^2 \right\rangle} + Q_2 \right]. \tag{76}$$

At last, we obtain (2) where

$$k_0 = \frac{5X_2}{2X_1} - \frac{X_5}{X_3} \frac{15\sqrt{\pi}}{16X_1} \tag{77}$$

and

$$k_1 = -\frac{5}{6} + \frac{5X_2}{6X_1} + \frac{X_4}{3X_3} - \frac{X_5}{X_3} \frac{5\sqrt{\pi}}{16X_1}. \tag{78}$$

Summary and numerical solution

The flow coefficient is given by (2) and (77)-(78) where

$$\begin{aligned} X_1 &= \int_0^\infty dx\, e^{-x^2} x^3 \left(x^2 - \frac{5}{2} \right) F_1(x) \\ X_2 &= \int_0^\infty dx\, e^{-x^2} x^3 F_1(x) \\ X_3 &= \int_0^\infty dx\, e^{-x^2} x^4 F_2(x) \\ X_4 &= \int_0^\infty dx\, e^{-x^2} x^4 F_4(x) \\ X_5 &= \int_0^\infty dx\, e^{-x^2} x^4 F_F(x) \end{aligned} \tag{79}$$

and the 4 Spitzer problems we must solve are

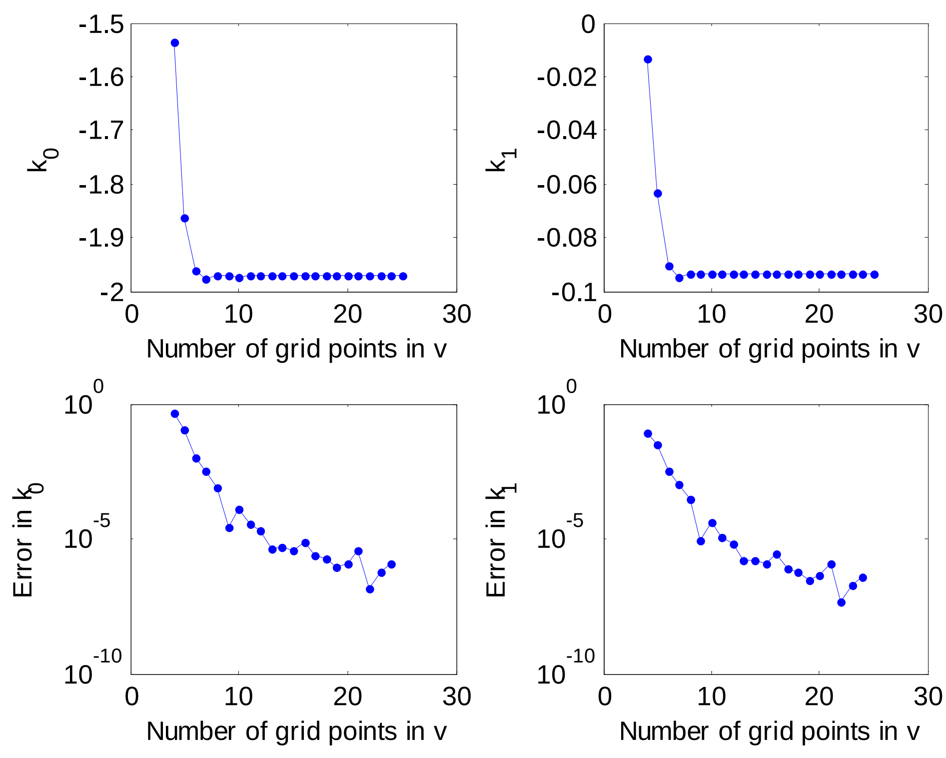
$$\frac{1}{\nu_{ii}} C\left\{F_1(x) f_0 \xi\right\} = f_0\left(x^2 - \frac{5}{2}\right) x\xi, \tag{80}$$

$$\frac{1}{\nu_{ii}} C\left\{F_2(x) f_0 L_2(\xi)\right\} = L_2(\xi) f_0 x^2, \tag{81}$$

$$\frac{1}{\nu_{ii}} C\left\{F_4(x) f_0 L_2(\xi)\right\} = L_2(\xi) f_0 x^4, \tag{82}$$

$$\frac{1}{\nu_{ii}} C\left\{F_F(x) f_0 L_2(\xi)\right\} = L_2(\xi) f_0 x F_1(x). \tag{83}$$

The Spitzer functions and integrals were computed using the matlab program m20131019_01_solveSpitzerProblemsForPfirschSchluterRegimeFlow.m, using the discretization techniques described in Ref. [4]. Here we do not numerically solve the finite-collisionality problem (6), only the spatially-independent problems (80)-(83). Using 25 polynomials to reduce the variation in the coefficients to $< 10^{-5}$ between successive numbers of polynomials, the converged values of the flow coefficients are given by (5), where all digits shown are converged. The plots below illustrate the convergence. The "error" shown in the bottom two plots is the difference between the value and the value for 25 grid points.



References

[1] Hazeltine, Phys. Fluids **17**, 961 (1974).

[2] Catto & Simakov, Phys. Plasmas **16**, 042503 (2009).

[3] Pusztai & Catto, Plasma Phys. Controlled Fusion **52**, 075016 (2010).
[4] Landreman & Ernst, J. Comp. Phys. **243**, 130 (2013).