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Another Boozer-coordinate-free motivation for quasisymmetry

We would like any confinement device to have the following omnigenity property: the ψ coordinate of each trapped particle has no net change per bounce. Equivalently, $\Delta\psi = 0$ where

$$\Delta\psi = \psi_{final} - \psi_{initial} = \oint \frac{d\psi}{dt} dt = \oint \mathbf{v}_d \cdot \nabla\psi \; dt . \tag{1}$$

Here, $\oint$ indicates an integral over a bounce, holding the magnetic moment $\mu = m\upsilon_\perp^2/(2B)$ and energy $m\upsilon^2/2$ fixed. Rather than integrate over the actual orbit (which has a nonzero width), we take (1) to be an integral along a field line, which is the same aside from a correction of order ρ_*, and which is much easier to work with mathematically.

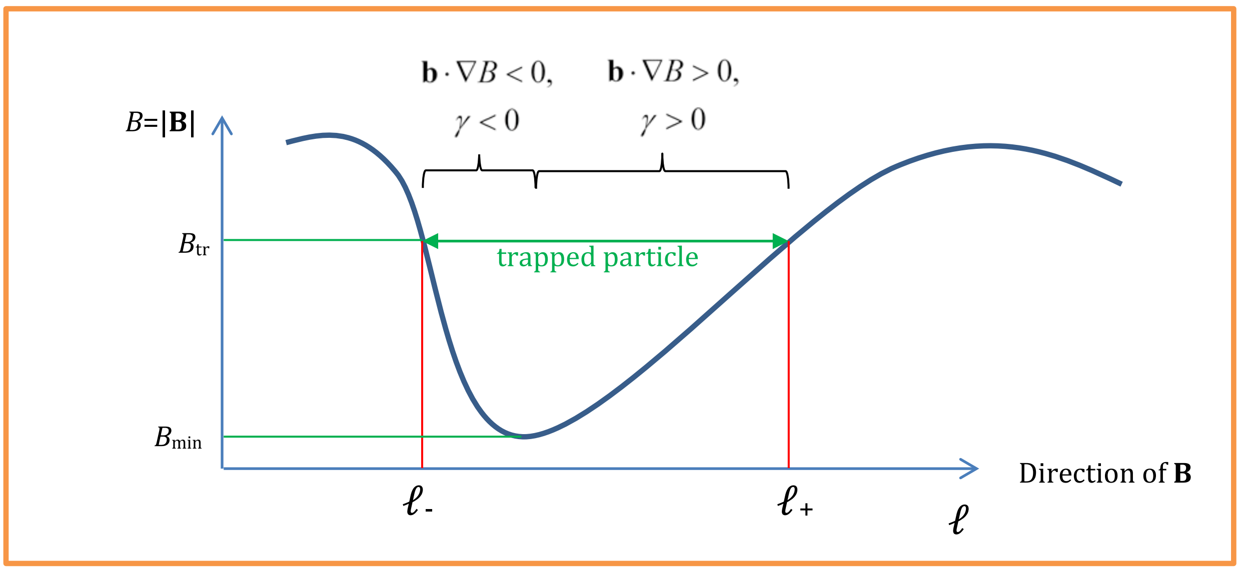
Before considering stellarators, let us examine in detail why (1) is satisfied in a tokamak. First, we notice

$$\Delta\psi = \oint (\mathbf{v}_d \cdot \nabla\psi) dt = 2\int_{\ell_-}^{\ell_+} (\mathbf{v}_d \cdot \nabla\psi) \frac{d\ell}{|\upsilon_\||} \tag{2}$$

where ℓ_- and ℓ_+ are the two bounce points. Changing the variable of integration to $B = |\mathbf{B}|$,

$$\Delta\psi = 2\sum_\gamma \int_{B_{\min}}^{B_{tr}} (\mathbf{v}_d \cdot \nabla\psi) \frac{dB}{|\upsilon_\|| |\mathbf{b}\cdot\nabla B|} = 2\sum_\gamma \gamma \int_{B_{\min}}^{B_{tr}} (\mathbf{v}_d \cdot \nabla\psi) \frac{dB}{|\upsilon_\|| \mathbf{b}\cdot\nabla B} \tag{3}$$

where $B_{tr} = m\upsilon^2/(2\mu)$ is the value of B at which the particle is trapped, and where $\gamma = sign(\mathbf{b}\cdot\nabla B)$ is the "branch": $\gamma = +1$ if our location relative to $B_{\min}$ is *parallel* to $\mathbf{B}$, and $\gamma = -1$ if our location relative to $B_{\min}$ is *antiparallel* to $\mathbf{B}$. The sum over γ is needed because each value of B could correspond to one of two different ℓ, as shown by the following figure. As with any change of variables, the *absolute value* of the Jacobian arises, and in the case of (3) this is $1/|\mathbf{b}\cdot\nabla B|$.



Next, using $\mathbf{B}\times\boldsymbol{\kappa}\cdot\nabla\psi = B^{-1}\mathbf{B}\times\nabla B\cdot\nabla\psi$ (which is true for *any* MHD equilibrium), we obtain

$$\Delta\psi = \frac{2mc}{Ze}\sum_{\gamma}\gamma\int_{B_{\min}}^{B_{\mathrm{tr}}}\left(\upsilon_{\parallel}^{2}+\frac{\upsilon_{\perp}^{2}}{2}\right)\frac{1}{\left|\upsilon_{\parallel}\right|B^{2}}\frac{\mathbf{B}\times\nabla B\cdot\nabla\psi}{\mathbf{B}\cdot\nabla B}dB. \tag{4}$$

In an axisymmetric MHD equilibrium, $\mathbf{B}=\nabla\phi\times\nabla\psi+F(\psi)\nabla\phi$ where ϕ is the toroidal angle and $F(\psi)=RB_{tor}$ is a flux function. Therefore $\mathbf{B}\times\nabla\psi = F\mathbf{B}-R^{2}B^{2}\nabla\phi$, so

$$\frac{\mathbf{B}\times\nabla B\cdot\nabla\psi}{\mathbf{B}\cdot\nabla B} = -F(\psi). \tag{5}$$

Then we can pull this factor outside the integral in (4):

$$\Delta\psi = -\frac{2mcF}{Ze}\sum_{\gamma}\gamma\int_{B_{\min}}^{B_{\mathrm{tr}}}\left(\upsilon_{\parallel}^{2}+\frac{\upsilon_{\perp}^{2}}{2}\right)\frac{1}{\left|\upsilon_{\parallel}\right|B^{2}}dB. \tag{6}$$

It turns out that the remaining integral can actually be done, using the relation $\upsilon^{2}=\upsilon_{\parallel}^{2}+(2\mu B/m)$ and recalling that μ and υ are fixed. However, we do not even need to do the integral, because the result evidently cannot depend on the branch (only on $B_{\min}$, B_{tr}, υ, and μ), and so the result will vanish in the γ sum. Physically, the radial drift in the $\gamma>0$ branch is equal and opposite to the radial drift in the $\gamma<0$ branch.

In the above argument, we only used axisymmetry in one small way: writing the left-hand side of (5) as a flux function, so that function could be pulled outside the integral. It did not even matter that the ratio equaled RB_{tor}. In fact, it did not even matter that the ratio was a flux function – only that it was branch-independent! Thus, if we can find any *nonaxisymmetric* field in which

$$\frac{\mathbf{B}\times\nabla B\cdot\nabla\psi}{\mathbf{B}\cdot\nabla B} = Y(\psi,B) \tag{7}$$

for some function Y, then (4) would become

$$\Delta\psi = \frac{2mc}{Ze}\sum_{\gamma}\gamma\ L(B_{\min},\upsilon,\mu) \tag{8}$$

where

$$L(B_{\min},\upsilon,\mu) = \int_{B_{\min}}^{B_{\mathrm{tr}}}\left(\upsilon_{\parallel}^{2}+\frac{\upsilon_{\perp}^{2}}{2}\right)\frac{1}{\left|\upsilon_{\parallel}\right|B^{2}}Y(\psi,B)\,dB. \tag{9}$$

Then the sum over γ in (8) immediately vanishes because L is independent of γ. Thus, any magnetic field with the property (7) is omnigenous: $\Delta\psi=0$.

As shown in Ref. [1], the quasisymmetry condition $B=B(\psi,\ M\theta-N\zeta)$ for any integers M and N is satisfied if and only if

$$\frac{\mathbf{B}\times\nabla B\cdot\nabla\psi}{\mathbf{B}\cdot\nabla B} = Y(\psi). \tag{10}$$

This condition, that the ratio is a flux function, is a stricter condition than (7). Therefore quasisymmetry implies omnigenity.

I'm not sure if (7) is as general as omnigenity: here we have only proved that (7) is sufficient for omnigenity, not that (7) is necessary.

[1] Helander and Simakov, **PRL** 101, 145003 (2008).