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Equivalence of symmetry in Boozer and Hamada coordinates

The following derivation is based on Appendix A of Sugama and Nishimura, *Phys. Plasmas* **9**, 4637 (2002).

Hamada angles θ_H and ζ_H can be defined by

$$\mathbf{B} = \nabla\psi \times \nabla\theta_H + \iota\nabla\zeta_H \times \nabla\psi \tag{1}$$

with

$$\nabla\psi \cdot \nabla\theta_H \times \nabla\zeta_H = \frac{4\pi^2}{V'} \tag{2}$$

where $2\pi\psi$ is the toroidal flux, V is the volume enclosed by a flux surface, and the prime denotes $d/d\psi$.

In contrast, Boozer angles θ_H and ζ_H can be defined by

$$\mathbf{B} = \nabla\psi \times \nabla\theta_B + \iota\nabla\zeta_B \times \nabla\psi \tag{3}$$

and

$$\mathbf{B} = B_\psi^B \nabla\psi + B_\theta^B \nabla\theta_B + B_\zeta^B \nabla\zeta_B \tag{4}$$

with B_θ^B and B_ζ^B flux functions. Note that

$$\left\langle B^2 \right\rangle = \frac{1}{V'}\int_0^{2\pi} d\theta_B \int_0^{2\pi} d\zeta_B \frac{B^2}{\nabla\psi \cdot \nabla\theta_B \times \nabla\zeta_B}, \tag{5}$$

where $\langle\ \rangle$ denotes a flux surface average, and

$$V' = \int_0^{2\pi} d\theta_B \int_0^{2\pi} d\zeta_B \frac{1}{\nabla\psi \cdot \nabla\theta_B \times \nabla\zeta_B}. \tag{6}$$

As

$$\nabla\psi \cdot \nabla\theta_B \times \nabla\zeta_B = \frac{B^2}{B_\zeta^B + \iota B_\theta^B}, \tag{7}$$

then

$$\left\langle B^2 \right\rangle = \frac{4\pi^2}{V'}\left(B_\zeta^B + \iota B_\theta^B\right) \tag{8}$$

and

$$\nabla\psi \cdot \nabla\theta_B \times \nabla\zeta_B = \frac{4\pi^2}{V'}\frac{B^2}{\left\langle B^2 \right\rangle}. \tag{9}$$

Transformations between magnetic coordinates

Suppose the transformation from one system to the other was written as

$$\theta_H = \theta_B + F\left(\psi, \theta_B, \zeta_B\right) \tag{10}$$

$$\zeta_H = \zeta_B + G\left(\psi, \theta_B, \zeta_B\right) \tag{11}$$

where F and G are periodic in both the poloidal and toroidal angles. Equating (3) and (1),

$$\nabla\psi \times \nabla F + \iota\nabla G \times \nabla\psi = 0. \tag{12}$$

The $\nabla\theta_B$ component of this equation tells us $\partial F / \partial\zeta_B = \iota\, \partial G / \partial\zeta_B$, so upon integrating,

$$F = \iota G + y\left(\psi, \theta_B\right). \tag{13}$$

Here and throughout this document, $\partial / \partial\zeta_B$ holds θ_B fixed, $\partial / \partial\theta_B$ holds ζ_B fixed, $\partial / \partial\zeta_H$ holds θ_H fixed, and $\partial / \partial\theta_H$ holds ζ_H fixed. The $\nabla\zeta_B$ component of (12) implies $\partial F / \partial\theta_B = \iota\, \partial G / \partial\theta_B$, so

$$F = \iota G + w\left(\psi, \zeta_B\right). \tag{14}$$

Comparing (13) with (14), F must equal ιG plus a flux function, so

$$\theta_H = \theta_B + \iota G \tag{15}$$

$$\zeta_H = \zeta_B + G + a\left(\psi\right). \tag{16}$$

(The $a\left(\psi\right)$ term in (16) is not crucial, since adding a flux function to any of the angular magnetic coordinates results in an equally valid coordinate within the same coordinate system.)

Useful identities

Applying $\mathbf{B}\cdot\nabla$ to (16), we obtain

$$\nabla\psi\times\nabla\theta_H\cdot\nabla\zeta_H = \nabla\psi\times\nabla\theta_B\cdot\nabla\zeta_B + \mathbf{B}\cdot\nabla G. \tag{17}$$

(The same result could be obtained by applying $\mathbf{B}\cdot\nabla$ to (15)). Noting (9) and (2), then (17) can be written

$$\frac{\partial G}{\partial\zeta_B} + \iota\frac{\partial G}{\partial\theta_B} = \frac{\left\langle B^2\right\rangle}{B^2} - 1. \tag{18}$$

Next, from the chain rule,

$$\frac{\partial G}{\partial\theta_H} = \frac{\partial\theta_B}{\partial\theta_H}\frac{\partial G}{\partial\theta_B} + \frac{\partial\zeta_B}{\partial\theta_H}\frac{\partial G}{\partial\zeta_B}. \tag{19}$$

By applying $\partial / \partial\theta_H$ to (15) and (16), we find $1 = \partial\theta_B / \partial\theta_H + \iota\partial G / \partial\theta_H$ and $0 = \partial\zeta_B / \partial\theta_H + \partial G / \partial\theta_H$, so (19) implies

$$\frac{\partial G}{\partial\theta_H} = \left(1 - \iota\frac{\partial G}{\partial\theta_H}\right)\frac{\partial G}{\partial\theta_B} - \frac{\partial G}{\partial\theta_H}\frac{\partial G}{\partial\zeta_B}. \tag{20}$$

Rearranging,

$$\left(1 + \iota\frac{\partial G}{\partial\theta_B} + \frac{\partial G}{\partial\zeta_B}\right)\frac{\partial G}{\partial\theta_H} = \frac{\partial G}{\partial\theta_B}, \tag{21}$$

so recalling (18), then

$$\frac{\left\langle B^2\right\rangle}{B^2}\frac{\partial G}{\partial\theta_H} = \frac{\partial G}{\partial\theta_B}. \tag{22}$$

A similar calculation gives

$$\frac{\left\langle B^2\right\rangle}{B^2}\frac{\partial G}{\partial\zeta_H} = \frac{\partial G}{\partial\zeta_B}. \tag{23}$$

Next, applying $\partial / \partial\theta_B$ to (18) and commuting the derivatives on the left-hand side,

$$\left[\frac{\partial}{\partial\zeta_B}+\iota\!\!\!-\frac{\partial}{\partial\theta_B}\right]\frac{\partial G}{\partial\theta_B}=-\frac{2\left\langle B^2\right\rangle}{B^3}\frac{\partial B}{\partial\theta_B}\,. \tag{24}$$

Recalling that $\mathbf{B}\cdot\nabla=\nabla\psi\times\nabla\theta_B\cdot\nabla\zeta_B\left[\left(\partial/\partial\zeta_B\right)+\iota\!\!\!-\left(\partial/\partial\theta_B\right)\right]$, then (24) is equivalent to

$$\mathbf{B}\cdot\nabla\frac{\partial G}{\partial\theta_B}=-\frac{4\pi^2}{V'}\frac{2}{B}\frac{\partial B}{\partial\theta_B}\,. \tag{25}$$

We could have applied $\partial/\partial\zeta_B$ to (18) instead of $\partial/\partial\theta_B$, and so it is also true that

$$\mathbf{B}\cdot\nabla\frac{\partial G}{\partial\zeta_B}=-\frac{4\pi^2}{V'}\frac{2}{B}\frac{\partial B}{\partial\zeta_B}\,. \tag{26}$$

Boozer symmetry implies Hamada symmetry

Suppose $B=B\left(M\theta_B-N\zeta_B\right)$, i.e.

$$N\frac{\partial B}{\partial\theta_B}+M\frac{\partial B}{\partial\zeta_B}=0\,. \tag{27}$$

Then from (25)-(26),

$$\mathbf{B}\cdot\nabla\left(N\frac{\partial G}{\partial\theta_B}+M\frac{\partial G}{\partial\zeta_B}\right)=0\,. \tag{28}$$

It follows that

$$N\frac{\partial G}{\partial\theta_B}+M\frac{\partial G}{\partial\zeta_B}=A\left(\psi\right) \tag{29}$$

for some flux function $A\left(\psi\right)$. Integrating (29) in θ_B and ζ_B from 0 to 2π in both variables, we obtain $0=\left(2\pi\right)^2 A$, so $A=0$. Then applying (22) and (23),

$$N\frac{\partial G}{\partial\theta_H}+M\frac{\partial G}{\partial\zeta_H}=0\,. \tag{30}$$

Finally, we form

$$\begin{aligned}
N\frac{\partial B}{\partial\theta_H}+M\frac{\partial B}{\partial\zeta_H}&=N\left(\frac{\partial\theta_B}{\partial\theta_H}\frac{\partial B}{\partial\theta_B}+\frac{\partial\zeta_B}{\partial\theta_H}\frac{\partial B}{\partial\zeta_B}\right)+M\left(\frac{\partial\theta_B}{\partial\zeta_H}\frac{\partial B}{\partial\theta_B}+\frac{\partial\zeta_B}{\partial\zeta_H}\frac{\partial B}{\partial\zeta_B}\right)\\
&=N\left(\left[1-\iota\!\!\!-\frac{\partial G}{\partial\theta_H}\right]\frac{\partial B}{\partial\theta_B}-\frac{\partial G}{\partial\theta_H}\frac{\partial B}{\partial\zeta_B}\right)+M\left(-\iota\!\!\!-\frac{\partial G}{\partial\zeta_H}\frac{\partial B}{\partial\theta_B}+\left[1-\frac{\partial G}{\partial\zeta_H}\right]\frac{\partial B}{\partial\zeta_B}\right)\\
&=\left(N\frac{\partial B}{\partial\theta_B}+M\frac{\partial B}{\partial\zeta_B}\right)-\iota\!\!\!-\frac{\partial B}{\partial\theta_B}\left(N\frac{\partial G}{\partial\theta_H}+M\frac{\partial G}{\partial\zeta_H}\right)-\frac{\partial B}{\partial\zeta_B}\left(N\frac{\partial G}{\partial\theta_H}+M\frac{\partial G}{\partial\zeta_H}\right).
\end{aligned} \tag{31}$$

The first equality above is the chain rule; to get from the first line to the second we have used the $\partial/\partial\theta_H$ and $\partial/\partial\zeta_H$ derivatives of (15) and (16), and the last line follows from algebraic rearrangement. The terms in parentheses all vanish in a field which satisfies (27) due to (30), and so the Boozer symmetry property (27) implies

$$N\frac{\partial B}{\partial\theta_H}+M\frac{\partial B}{\partial\zeta_H}=0\,. \tag{32}$$

Thus, symmetry in Boozer angles implies symmetry in Hamada angles.