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H-theorem

Recall that the rate of entropy density production in the distribution for species j is

$$\frac{\partial S_j}{\partial t} = -\int d^3v \ \ln f_j \ \sum_k C_{jk} \tag{1}$$

Let $\dot{S}_{jj}$ denote the contribution to this entropy production from the *nonlinear* Fokker-Planck operator for self-collisions:

$$\dot{S}_{jj} = -\int d^3v \ \ln f_j \ C_{jj} \tag{2}$$

(It would be silly to use a *linearized* operator before we knew we had a Maxwellian.)

Claim: $\dot{S}_{jj}$ vanishes only if f_j is a drifting Maxwellian.

Proof:

For the rest of this proof I'll drop the subscript on f_j. Plugging in the Landau form of the collision operator,

$$\begin{aligned}\dot{S}_{jj} &= \gamma \int d^3v \ln f \nabla_v \cdot \int d^3v' \nabla_g \nabla_g g \cdot \left[f' \nabla_v f - f \nabla_{v'} f' \right] \\ &= -\gamma \int d^3v \int d^3v' f' f \left[\nabla_v \ln f \right] \cdot \nabla_g \nabla_g g \cdot \left[\nabla_v \ln f - \nabla_{v'} \ln f' \right] \\ &= -\gamma \int d^3v \int d^3v' f' f \left[\nabla_{v'} \ln f' \right] \cdot \nabla_g \nabla_g g \cdot \left[\nabla_{v'} \ln f' - \nabla_v \ln f \right]\end{aligned} \tag{3}$$

where in the last line we have switched primed and unprimed variables. Averaging the last two lines, we get

$$\dot{S}_{jj} = -\frac{\gamma}{2} \int d^3v \int d^3v' f' f \left[\nabla_v \ln f - \nabla_{v'} \ln f' \right] \cdot \nabla_g \nabla_g g \cdot \left[\nabla_v \ln f - \nabla_{v'} \ln f' \right] \tag{4}$$

Letting $\mathbf{H} = \nabla_v \ln f - \nabla_{v'} \ln f'$, then

$$\dot{S}_{jj} = -\frac{\gamma}{2} \int d^3v \int d^3v' \frac{f' f}{g^3} \left\{ g^2 H^2 - \left[\mathbf{g} \cdot \mathbf{H} \right]^2 \right\} \tag{5}$$

If $\dot{S}_{jj}$ vanishes, then the quantity in curly braces must vanish for all $\mathbf{v}$ and for all $\mathbf{v}'$. Using the Schwartz inequality for $\mathbb{R}^3$, it must then be that

$$\mathbf{g} \,||\, \mathbf{H} \text{ for all } \mathbf{v} \text{ and for all } \mathbf{v}'. \tag{6}$$

At this point it is helpful to cast the problem into the notation of Sussman and Wisdom. As part of this notation, throughout this document I will use parentheses only to denote the input parameters to a function, not to denote the order of operations; for the latter I will use only square and curly braces. The statement (6) means that there exists a function $\lambda : \mathbb{R}^6 \rightarrow \mathbb{R}$ such that the following three equations are true for all $a,\ b,\ c,\ d,\ e,$ and h:

$$\left[a - d \right] \lambda\left(a,b,c,d,e,h \right) = F_0\left(a,b,c \right) - F_0\left(d,e,h \right), \tag{7}$$

$$\left[b - e \right] \lambda\left(a,b,c,d,e,h \right) = F_1\left(a,b,c \right) - F_1\left(d,e,h \right), \tag{8}$$

$$\left[c - f \right] \lambda\left(a,b,c,d,e,h \right) = F_2\left(a,b,c \right) - F_2\left(d,e,h \right) \tag{9}$$

where $F_i = \partial_i \ln f$. (Here I'm just using a, b, and c to represent the components of $\mathbf{v}$, and using d, e, and h to represent the components of $\mathbf{v}'$.) Considering (7) for the particular case that $a = d$, then

$$F_0\left(a,b,c\right) = F_0\left(a,e,h\right). \tag{10}$$

Therefore F_0 must be independent of its latter two inputs, so we can write

$$F_0\left(a,b,c\right) = x\left(a\right) \tag{11}$$

where x is some $\mathbb{R} \to \mathbb{R}$ function. Repeating this procedure for (8) and (9), we find there must exist functions y and z such that

$$F_1\left(a,b,c\right) = y\left(b\right) \text{ and} \tag{12}$$

$$F_2\left(a,b,c\right) = z\left(c\right). \tag{13}$$

Using these results we can rewrite (7)-(9) as

$$\left[a-d\right]\lambda\left(a,b,c,d,e,h\right) = x\left(a\right) - x\left(d\right), \tag{14}$$

$$\left[b-e\right]\lambda\left(a,b,c,d,e,h\right) = y\left(b\right) - y\left(e\right), \tag{15}$$

$$\left[c-h\right]\lambda\left(a,b,c,d,e,h\right) = z\left(c\right) - z\left(h\right). \tag{16}$$

Note that if $a \neq d$, then (14) can be written such that the right-hand side is independent of $b, c, e,$ and h:

$$\lambda\left(a,b,c,d,e,h\right) = \left[x\left(a\right) - x\left(d\right)\right] / \left[a-d\right] \tag{17}$$

Thus, λ must be independent of its 2nd, 3rd, 5th, and 6th inputs, at least outside of the hyperplane in its input space defined by $a = d$. If we require that f be smooth, then since $x\left(a\right) = \left[\partial_0 \ln f\right]\left(a,b,c\right)$ is a derivative of the distribution function, x will be differentiable. In this case, in the limit $d \to a$, the right-hand side of (17) converges to $\left[Dx\right]\left(a\right)$, and so we can write

$$\lambda\left(a,b,c,a,e,h\right) = \left[Dx\right]\left(a\right). \tag{18}$$

The right-hand side is independent of $b, c, e,$ and h, and so λ must be independent of these inputs on the hyperplane $a = d$ as well.

The argument above can be repeated using (15) to show that λ is also independent of its 1st and 4th inputs. Therefore λ is a constant.

We can now write (14) as

$$x\left(a\right) - a\lambda = x\left(d\right) - d\lambda. \tag{19}$$

Since this equation is true for all a and d, each side of this equation is a constant (i.e. independent of $a, b, c, d, e,$ and h.) We name this constant x_0. Recalling the definition of x in terms of the distribution function, then

$$\left[\partial_0 \ln f\right]\left(a,b,c\right) = a\lambda + x_0. \tag{20}$$

Integrating in a, then

$$\ln f\left(a,b,c\right)=\frac{\lambda}{2}a^{2}+x_{0}a+A\left(b,c\right) \tag{21}$$

where $A\left(b,c\right)$ is the "constant of integration." Repeating the last few steps using (15) and (16) instead of (14), we find that there must exist constants y_0 and z_0 and functions $B\left(a,c\right)$ and $C\left(a,b\right)$ such that

$$\ln f\left(a,b,c\right)=\frac{\lambda}{2}b^{2}+y_{0}b+B\left(a,c\right) \tag{22}$$

and

$$\ln f\left(a,b,c\right)=\frac{\lambda}{2}c^{2}+z_{0}c+C\left(a,b\right). \tag{23}$$

The only way that (21), (22), and (23) can simultaneously be true is if

$$\ln f\left(a,b,c\right)=\frac{\lambda}{2}\left[a^{2}+b^{2}+c^{2}\right]+x_{0}a+y_{0}b+z_{0}c+w \tag{24}$$

where w is another constant. Now recognizing $\lambda\rightarrow -M/T$, $x_{0}=MV_{x}/T$, etc., we see that f is a Maxwellian.