Fidelity and Leakage of Josephson Qubits

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The unit of quantum information is the qubit, a vector in a two-dimensional Hilbert space. On the other hand, quantum hardware often operates in two-dimensional subspaces of vector spaces of higher dimensionality. The presence of higher quantum states may affect the accuracy of quantum information processing. In this Letter we show how to cope with quantum leakage in devices based on small Josephson junctions. While the presence of higher charge states of the junction reduces the fidelity during gate operations we demonstrate that errors can be minimized by appropriately designing and operating the gates.

The most widely accepted paradigm of quantum computation describes quantum information processing in terms of quantum gates whose input and output are two-state quantum systems called qubits [1]. Quantum computation (QC) is performed by means of a controllable unitary evolution of the qubits [2]. Because of the intrinsic quantum parallelism, some problems which are intractable on classical computers can be solved efficiently by using quantum algorithms. Probably the most striking example is the factorization of large numbers [3].

Parallel to the development of the theory of quantum information, there has been an increasing interest in finding physical systems where quantum computation could be implemented. In an (almost) ideal situation one should identify a suitable set of two-level systems (sufficiently decoupled from any source of decoherence [4]) with some controllable couplings among them needed to realize single-qubit and two-qubit operations. These requirements are sufficient to implement any computational task [5]. Various physical systems have been suggested for the implementation of quantum algorithms, e.g., ions traps [6], QED cavities [7], and NMR [8]. The quest for large scale integrability and flexibility in the design has very recently stimulated an increasing interest in the field of nanostructures. Until now promising proposals are based on small-capacitance Josephson junctions [9–13], coupled quantum dots [14,15], neutral atoms in optical lattices [16], and phosphorus dopants in silicon crystals [17]. The experiments on the superposition of charge states in Josephson junctions [18,19] and the recent achievements in controlling the coherent evolution of quantum states in a Cooper pair box [20] render superconducting nanocircuits interesting candidates to implement solid state quantum computers.

Physical realizations of QC are never completely decoupled from the environment. Since decoherence will ultimately limit the performance of a quantum computer, a lot of attention is being devoted to this problem. Besides decoherence, for each proposed scheme a detailed analysis of the errors induced by the gate operations themselves is crucial in order to assess their reliability and the feasibility of fault-tolerant quantum computation [21,22]. Errors may occur due to a variety of reasons. An obvious example is fluctuations in the control parameters of the gate which act as a random noise and thus affect the unitarity of the time evolution of the computational degrees of freedom. Alternatively, gate operations can change the coupling of the qubits to the environment (even if this coupling is negligible during storage periods) thereby enhancing decoherence. All these error sources can be analyzed by properly modeling the qubit-environment coupling. However, there are errors which are not due to (or cannot be described in terms of) the action of an external environment.

In this Letter we consider one (intrinsic) source of error in gate operations which is common to several of the proposed solid state implementations, the quantum leakage. It occurs when the computational space is a subspace of a larger Hilbert space. This is the case, e.g., when the information is encoded in trapped ions or in charge (or flux) states of devices based on Josephson junctions (or SQUIDS). We start by introducing a general scheme to characterize the leakage and then we focus on devices based on small-capacitance tunnel junctions. The effects of states outside the computational space have been investigated for ion trap quantum computers [23]. There the authors gave estimates of the number of operations before decay processes induce dissipative transitions outside the computational space. Here we consider the effect of non-computational states on the gate fidelity, and we compute the probability of leakage. Noncomputational states affect the gate dynamics even in the absence of dissipative processes.

Our analysis applies to the situation illustrated in Fig. 1. The two low-energy states constitute the computational Hilbert space. The system, however, can leak out to the higher states. If the energy difference between the low-lying and the excited states is large compared to the other energy scales of the system (as in Refs. [9–13]) the probability to leak out is small. One might wonder whether it is necessary to discuss this effect at all. As we will see...
The consequences of leakage are more severe than a simple estimate of energy scales might suggest. The presence of states outside the computational space modifies the time evolution of the qubit states compared to the idealized design.

The ideal unitary gate operation $U_I$ is obtained by switching on a suitable Hamiltonian $H_I$ which couples the desired computational states in a controlled way for a time $t_0$. By choosing $t_0$ one can implement the desired gate operation. In reality, however, the dynamics of the system is governed by a unitary operator $U_R$ which acts on the full Hilbert space. Since information is being processed within the computational subspace the output is related to the input state via the map $\Pi U_R(t) \Pi$, where $\Pi$ is the projection operator on the computational space. One is interested in optimizing the real gate operation in order to get as close as possible to the ideal $U_I$. In general the “best” operation may require a time $t \neq t_0$ as all the system eigenenergies are modified by the states outside the computational subspace. Therefore we use the time $t$ as a parameter to optimize the given computational step. We characterize the performance of real gates by the fidelity $F$ and the probability of leakage $L(t)$ defined as

$$F = 1 - \frac{1}{2} \min_\psi \|U_I(t_0) - \Pi U_R(t) \Pi \|,$$

$$L(t) = 1 - \min_\psi \langle \psi | U_R(t) \Pi U_R(t) | \psi \rangle.$$  

In Eq. (1) we make use of the operator norm defined as $\|D\| = \sup_\psi |D|\psi| = \sup_\psi \sqrt{\langle \psi |D^+D| \psi \rangle}$ over the vectors $|\psi\rangle : \langle \psi | \psi \rangle = 1$ of the computational subspace. This definition implies that $\|D\| = \sqrt{\lambda_M}$, where $\lambda_M$ is the biggest eigenvalue of $D^+D$. As in the case of the minimal fidelity [24] this definition gives estimates for the worst case. The definition given in Eq. (1) can therefore be regarded as a prescription how to optimize the gate design [note that the fidelity defined in Eq. (1) does not depend on the time $t$].

As mentioned before, the existence of states other than the computational ones has two main consequences on the qubit dynamics. There is a nonzero probability of leakage, measured by $L(t)$, and a modification of the eigenenergies and eigenstates of the real system. The latter effect turns out to be an important source of gate errors.

In order to study the phenomena related to leakage quantitatively we apply Eqs. (1) and (2) to Josephson junction qubits in the charge regime as proposed in Refs. [9,11]. A similar analysis can be carried out, with appropriate changes of parameters, for all other cases where leakage is present. In Refs. [9,11] the qubit is implemented using nanocircuits of Josephson junctions. The corresponding Hamiltonian for one- and two-qubit operations can be written as

$$H_R = \sum_{i=1,2} \left[ E_{ch}(n_i - n_{x,i})^2 - E_J \cos \phi_i \right] + E_L(\sin \phi_1 + \sin \phi_2)^2.$$  

In the first term $E_{ch}$ is the charging energy. The second and the third terms represent the Josephson tunneling (associated with the energy $E_J$) and the inductive coupling of strength $E_L$ [25] which bring about single- and two-qubit operations. Both $E_J$ and $E_L$ are assumed to be much smaller than the charging energy. The offset charge $n_{x,i}$ can be controlled by an external gate voltage. The phases $\phi_i$ and the number of Cooper pairs $n_i$ are canonically conjugate variables $[\phi_i, n_j] = i \delta_{ij}$ [26].

At temperatures much lower than the charging energy, for $n_{x,i} \sim 1/2$ the two charge states $n_i = 0, 1$ are nearly degenerate. They represent the states $|0\rangle$, $|1\rangle$ of the qubit (see Fig. 1). In the computational Hilbert space the ideal evolution of the system is governed by the Hamiltonian,

$$H_I = \sum_{i=1,2} \Delta E_{ch,i} \sigma_{z,i} - \frac{E_J}{2} \sigma_{x,i} - \frac{E_L}{2} \sigma_{y,1} \sigma_{y,2},$$  

where $\sigma$ are Pauli matrices and $\Delta E_{ch,i} = E_{ch}(n_{x,i} - 1/2)$. The different time evolution due to $H_R$ and to $H_I$ causes an error in the gate operation. We note that leakage is also present during idle periods of the gates. However, here we discuss only the errors during gate operations.

One-bit gate ($E_L = 0$).—Single-qubit gate operations can be implemented, e.g., by suddenly switching the offset charge to the degeneracy point $n_x = 1/2$ where the charge states $|0\rangle$ and $|1\rangle$ are strongly mixed by the Josephson coupling [27]. Whereas in the ideal setup this coupling mixes only the states $|0\rangle$ and $|1\rangle$, in the real qubit all charge states are involved.
The evolution in the computational subspace for a time interval \( t \) is described by the operator \( \hat{\Pi} U_R(t) \hat{\Pi} = \sum e^{-iE_n t} \hat{\Pi} |\Phi_n\rangle \langle \Phi_n| \hat{\Pi} \),
(5)
where \( \hat{\Pi} = |0\rangle \langle 0| + |1\rangle \langle 1| \) is the projector on the computational subspace and \( |\Phi_n\rangle \) are the eigenstates with energies \( E_n \) of the Hamiltonian \( H_R \) (here \( |\Phi_n\rangle \) can be expressed in terms of Mathieu functions). By evaluating the leakage according to Eq. (2) we obtain
\[
\mathcal{L}(t) = 1 - \min_{m} | \langle \Phi_n | m \rangle e^{-iE_n t} |^2 \\
\sim \frac{E_n^2}{8E_{ch}^2} [1 - \min_{m} \cos(2E_{ch} \pm E_J/2)t].
\]
(6)
The order of magnitude \((E_J/E_{ch})^2\) can be understood immediately by regarding the coupling to higher charge states as a perturbation to the ideal system of Eq. (4).

The fidelity has to be limited by the leakage since it describes the length of the projection of the true state at time \( t \) onto the ideal state at \( t_0 \). There is another effect contributing to the loss of fidelity: the presence of higher charge states renormalizes the energy eigenvalues thus leading to a frequency mismatch between ideal and real time evolution. However, due to the symmetry of the system and the fact that \( E_J \) is the only coupling energy to the states outside the computational subspace, there is a simple way to cure this problem. Let us consider a \( \pi \) rotation. The optimal gate is obtained by changing the operation time to \( t_0^* = \pi/\Delta E \), where \( \Delta E \) is the energy splitting between the two lowest eigenstates (as opposed to the time \( t_0 = \pi/E_J \) in the ideal system). The value of the fidelity is then given by
\[
\mathcal{F} = 1 - \frac{1}{2} \left| \sum_{m=0,1} \langle 0 | \Phi_n \rangle \langle \Phi_n | m \rangle e^{-iE_n t_0^*} - i \right| \\
\sim 1 - \frac{\frac{E_J^2}{2E_{ch}^2}}{32} \sqrt{2 + 2|\sin(2\pi E_{ch}/E_J)|}.
\]
(7)

We mention that the error accumulates linearly with the number of operations. For typical parameters of Josephson junctions \( E_J/E_{ch} \sim 0.02 \) one finds that after about \( 10^4 \) operations the loss of fidelity becomes of order unity.

**Two-bit gate (\( E_L \neq 0 \)).**—Among the many possibilities for the elementary two-qubit operation, choosing a particular one may be a nontrivial step in the course of implementing quantum hardware. Because of universality of quantum computation [5] one is free to use any generic \( 4 \times 4 \) unitary matrix as a two-qubit gate. From our point of view, a choice is optimal if it avoids errors stemming from a discrepancy of the ideal gate and the way of its implementation. Therefore, in the following we assume that the Hamiltonian as introduced in Eq. (4),
\[
H_I = \begin{pmatrix}
2\Delta E_{ch} & -E_J/2 & -E_J/2 & E_L/2 \\
-E_J/2 & 0 & -E_L/2 & -E_J/2 \\
-E_J/2 & -E_L/2 & 0 & -E_J/2 \\
E_L/2 & -E_J/2 & -E_J/2 & 2\Delta E_{ch}
\end{pmatrix}
\]
(8)
generates the ideal two-bit gate. In Eq. (8) we have used the basis \{\(00\), \(01\), \(10\), \(11\)\} (which is obtained as the direct product of the states introduced previously). The typical scale for the operation time \( t_0 \) is on the order of \( 1/E_L \).

In complete analogy with the one-bit gate, we find that the leakage is of the same order for the two-qubit operation:
\[
\mathcal{L} \sim \max \left\{ \left( \frac{E_J}{E_{ch}} \right)^2, \left( \frac{E_L}{E_{ch}} \right)^2 \right\}
\]
(the numerical coefficient is, in general, larger than in the one-bit case because there are more charge states outside the computational subspace directly coupled to the qubit states either by \( E_L \) or \( E_J \)).

The situation for the fidelity, however, is different. In order to estimate \( \mathcal{F} \) we consider a perturbative expansion of \( D^\dagger D \), where \( D = U_I(t_0) - \hat{\Pi} U_R(t) \hat{\Pi} \) up to second order in \( E_J/E_{ch}, E_L/E_{ch} \). The eigenvalues of this matrix have the form \( 2 - 2 \cos(E_{ct} - E_n^{(0)}(t_0)) + 2n \) order terms (here, \( E_n \) and \( E_n^{(0)} \) are the eigenvalues of \( H_R \) and \( H_I \), respectively). It turns out that due to the presence of several energy scales the frequency mismatch between real and ideal time evolution cannot be compensated for by adjusting the operation period. The leading terms of the fidelity can be written as
\[
\mathcal{F} \approx 1 - \frac{1}{2} \left( a \frac{E_J^2}{E_L E_{ch}} + b \frac{E_L}{E_{ch}} \right),
\]
(9)
where \( a \) and \( b \) are coefficients which depend on the particular choice of \( n_{x,J} \) and \( t_0 \). In Fig. 2 we show the numerical results for \( n_x = 1/4 \) and \( t_0 = \pi/E_L \). The loss of fidelity [the term in parentheses in Eq. (9)] is proportional to \( t_0 \). The maximum (the best operation one can achieve) scales linearly with \( E_J/E_{ch} \). This should be contrasted with the one-bit case where it scales quadratically. For the gate operation and for the junction parameters used in Fig. 2, the number of two-bit operations is of the order of \( 10^2 \) before fidelity is lost completely.
We mention that we have chosen the definitions for the leakage and the fidelity describing the “worst case” in order to avoid a dependence of the discussion on the preparation of the initial state. One could wonder whether the “generic case” is much more robust with respect to leakage. It is easy to convince oneself by checking various choices of initial states that the loss of fidelity is indeed on the order of the worst case estimates.

In conclusion, starting from given gate operations we have discussed their optimal implementation in real systems. We have shown that leakage limits the number of operations which can be performed reliably both for one- and two-qubit gates. For one-bit gates, one can correct leakage errors by changing the operation time. We have pointed out that with respect to fidelity it may be appropriate to choose the elementary two-qubit gate as it is determined by the implementation. Figure 2 shows the central result of this work: although leakage causes an inevitable loss of fidelity for two-qubit operations, this loss can be minimized by an appropriate choice of the device parameters. For circuit parameters available with present day technology, the limit on the number of operations set by leakage is of the same order as that set by decoherence [9].

Finally, we mention that one can speculate about correction procedures for errors caused by leakage. It should be possible to check during the computation whether leakage and the fidelity describing the evolution of the system only if it is outside the computational subspace. One can imagine to realize a low sensitivity SET transistor which is able to measure the system only if the charge is outside a specified window.

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[25] The coupling $E_L$ is also proportional to the Josephson coupling energy. For our purposes we can assume that $E_L$ is an independent energy scale in the problem.
[26] The model ignores quasiparticle tunneling which is suppressed if the temperature is much lower than the BCS superconducting gap.
[27] Alternatively, the qubit can be set to the degeneracy point adiabatically. In this paper we do not examine the various possibilities to realize one- and two-qubit gates. The qualitative results do not change although it may be more convenient, as far as leakage is concerned, to choose a particular scheme.