

CHAPTER 14

14.1) MEAN SPEEDS IN A MAXWELLIAN DISTRIBUTION

From (11),

$$\langle v^2 \rangle = \int_0^{\infty} v^2 P(v) dv = A \int_0^{\infty} \exp(-Mv^2/2\tau) v^4 dv \quad ,$$

where $A = (4/\pi^{1/2})(M/2\tau)^{3/2}$. Substitute variables,

$$u = Mv^2/2\tau \quad , \quad v^4 dv = \frac{1}{2}(2\tau/M)^{5/2} u^{3/2} du:$$

$$\langle v^2 \rangle = (2/\sqrt{\pi})(2\tau/M) \int_0^{\infty} \exp(-u) u^{3/2} du \quad .$$

The integral has the value $\Gamma(5/2) = (3/2)\Gamma(3/2) = (3/2) \times (1/2) \times \Gamma(1/2) = 3\pi^{1/2}/4$. Here $\Gamma(n)$ is the gamma function, see Appendix A, Eq. (A.6), and we have used the relations (A.7) and (A.8). With this result,

$$\langle v^2 \rangle = 3\tau/M \quad .$$

We saw in (3.65) that for an ideal gas atom, the thermal average kinetic energy $U = \langle U \rangle = 3\tau/2$. Because $\langle U \rangle = \frac{1}{2}M\langle v^2 \rangle$, this is the same as (118).

(b) For $v = v_{mp}$,

$$dP/dv = A [2v - (M/\tau)v^3] \exp(-Mv^2/2\tau) = 0 \quad .$$

This is satisfied when

$$v^2 = v_{mp}^2 = 2\tau/M \quad .$$

(c) Inserting (11) into the integral in (121):

$$\bar{c} = A \int_0^{\infty} v P(v) dv = A \int_0^{\infty} \exp(-Mv^2/2\tau) v^3 dv .$$

With the same substitution as in (a):

$$v^3 dv = \frac{1}{2}(2\tau/M)^2 u du ,$$

$$\bar{c} = (2/\sqrt{\pi})(2\tau/M)^{\frac{1}{2}} \int_0^{\infty} \exp(-u) u du .$$

The integral has the value 1, hence

$$\bar{c} = (8\tau/\pi M)^{\frac{1}{2}} . \quad (121)$$

(d) Let θ be the angle between the vector \vec{v} and the z-direction. Then $|v_z| = |v| \times |\cos\theta|$. Because $P(v)$ is isotropic, the averages over the magnitudes and over the angles may be separated:

$$\langle |v_z| \rangle = \langle |v| \rangle \langle |\cos\theta| \rangle = \bar{c} \langle |\cos\theta| \rangle . \quad (S1)$$

The surface element of the unit sphere is $d\Omega = d\phi \sin\theta d\theta$. Hence

$$\begin{aligned} \langle |\cos\theta| \rangle &= \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} |\cos\theta| \sin\theta d\theta \\ &= \frac{1}{4\pi} \times 2\pi \times 2 \int_0^{\pi/2} \cos\theta \sin\theta d\theta = 1/2 . \end{aligned}$$

Insertion into (S1) and combination with (121) yields (123).

14.2) MEAN KINETIC ENERGY IN A BEAM

(a) Note first that the derivation given in the text for the flux density relation (95) made no assumptions about

the velocity distribution of the molecules other than that it is isotropic. The result is a purely geometric one that holds for each speed interval $(v, v+dv)$ separately. Consider now the concentration dn of molecules with speeds in this interval. If n is the total concentration, then $dn = nP(v)dv$. The flux density dJ_n of these atoms is then, from (95), with $P(v)$ from (11):

$$dJ_n = \frac{1}{4}vdn = \frac{n}{4}v^3 \exp(-Mv^2/2\tau) dv .$$

If we substitute variables,

$$Mv^2/2\tau = u \quad , \quad v^3 dv = (2\tau^2/M^2) u du \quad :$$

$$dJ_n = (\tau/2M) \exp(-u) u du \quad . \quad (S1)$$

The associated energy flux density is

$$dJ_\epsilon = \frac{1}{2}Mv^2 dJ_n = \tau u dJ_n = (\tau^2/2M) \exp(-u) u^2 du \quad (S2)$$

The average kinetic energy is then

$$\langle \epsilon \rangle = \int dJ_\epsilon / \int dJ_n = \tau \int \exp(-u) u^2 du / \int \exp(-u) u du = 2\tau \quad . (S3)$$

(b) Let $\phi \ll 1$ be a small angle. Then, $f = \pi\phi^2/4\pi = \phi^2/4$ is the fraction of atoms whose velocities make an angle less than ϕ relative to the beam axis. If ϕ is small enough, the velocity component along the beam axis is essentially the same as the overall speed. Hence, the flux density of the atoms in the speed interval $(v, v+dv)$ is now given by

$$dJ_n = f(v)v dn = (2f\tau/M) \exp(-u) u du \quad ,$$

instead of (S1). The energy flux density differs from (S2) by the same factor f . The two factors cancel out in the calculation of $\langle \epsilon \rangle$, hence $\langle \epsilon \rangle = 2\tau$ again.

Comment. In case (b) all the kinetic energy is energy of motion along the beam axis, while for case (a) part of the energy is energy of motion perpendicular to the beam axis.

In fact, in the uncollimated beam, the distribution of velocities perpendicular to the beam axis is the same as inside the oven. Because of (119) this implies

$$\frac{1}{2}M\langle v_x^2 \rangle = \frac{1}{2}M\langle v_y^2 \rangle = \frac{1}{2}\tau \quad ,$$

that is, the average kinetic energy perpendicular to the beam axis is τ . Hence in the uncollimated beam the average kinetic energy of motion along the beam axis is only τ , not 2τ , in contrast to the collimated beam.

14.3) RATIO OF THERMAL TO ELECTRICAL CONDUCTIVITY

We write the electrical conductivity σ in the form

$$\sigma = qn\tilde{\mu} \quad , \quad (S1)$$

which follows by comparing the defining relation $\vec{J}_q = \sigma \vec{E}$ in Table 14.2 with the term $n\tilde{\mu}\vec{E}$ in the expression (49) for $\vec{J}_n = \vec{J}_q/q$. From (S1), (31) and (50):

$$k/\tau\sigma = \hat{C}_V/nq^2 \quad . \quad (S2)$$

The heat capacity of an ideal gas of N particles is, from (6.36), $C_V = 3N/2$. The heat capacity per unit volume is then

$$\hat{C}_V \equiv C_V/V = 3N/2V = 3n/2 \quad .$$

Insertion into (S2) yields (124).

14.4) THERMAL CONDUCTIVITY OF METALS

(a) The electron gas in copper is a highly degenerate Fermi gas. The heat capacity of such a gas was given in (7.37) and (7.38). In conventional units, from (7.38),

$$C_V = \gamma_0 T \quad , \quad \text{where} \quad \gamma_0 = \pi^2 Nk_B^2 / 2\tau_F \quad ,$$

where $\tau_F = \varepsilon_F$ follows from the electron concentration via (7.40). Numerical results of this calculation were given in Table 7.2; we simply take them over. For Cu at 300 K:

$$C_V = 0.50 \text{ mJ mol}^{-1}\text{K}^{-2} \times 300 \text{ K} = 150 \text{ mJ mol}^{-1}\text{K}^{-1} .$$

Note that this is the heat capacity per mole; to convert to values per unit volume, we multiply by the ratio

$$n/N_A = 8.5 \times 10^{22} \text{ cm}^{-3} / 6.0 \times 10^{23} \text{ mol}^{-1} = 0.14 \text{ mol cm}^{-3},$$

where n is the electron concentration given in Table 7.1, and N_A is Avogadro's constant. This leads to $\hat{C}_V = 21 \text{ mJ cm}^{-3}\text{K}^{-1}$.

(b) We saw in (31) that the thermal conductivity is given by $K = D\hat{C}_V$. If \hat{C}_V is in conventional units, K will also be in conventional units. The diffusivity of a Fermi gas was given in (97), $D = v_F^2 \tau_C / 3$. Our problem did not specify the relaxation time τ_C but the mean free path ℓ . For a degenerate Fermi gas it is easy to see that

$$\ell = v_F \tau_C : \tag{S1}$$

Only the electrons near the Fermi energy participate in transport processes. Their speed is v_F . If τ_C is the mean time between collisions, (S1) gives the mean distance traveled between collisions. Numerically, from the value of v_F in Table 7.1 and from the value of ℓ specified:

$$\tau_C = 4 \times 10^{-6} \text{ cm} / 1.56 \times 10^8 \text{ cm s}^{-1} = 2.6 \times 10^{-14} \text{ s} ,$$

$$D = v_F^2 \tau_C / 3 = 208 \text{ cm}^2 \text{ s}^{-1} ,$$

$$K = D\hat{C}_V = 4.4 \times 10^3 \text{ mJ cm}^{-1} \text{ s}^{-1} = 4.4 \text{ J cm}^{-1} \text{ s}^{-1} .$$

Note that the result (92) for D could also have been obtained from the classical value (81) if one simply substituted there

$$\bar{c} = v_F . \tag{S2}$$

(c) If we insert the various numerical values into (94),
 $\sigma = nq^2\tau_c/m$, we obtain

$$\begin{aligned}\sigma &= (8.5 \times 10^{22} \text{cm}^{-3})(1.6 \times 10^{-19} \text{C})^2(2.6 \times 10^{-14} \text{s}) / (0.91 \times 10^{-27} \text{g}) \\ &= 6.1 \times 10^{-2} \text{C}^2 \text{cm}^{-3} \text{g}^{-1} \text{s} = 6.1 \times 10^5 \Omega^{-1} \text{cm}^{-1}\end{aligned}$$

In the scientific literature, conductivities are almost always quoted in the units $\Omega^{-1} \text{cm}^{-1}$ (sometimes in $\Omega^{-1} \text{m}^{-1}$), from

$$\begin{aligned}1 \Omega^{-1} \text{cm}^{-1} &= 1 (\text{V/A})^{-1} \text{cm}^{-1} = 1 (\text{A/cm}^2) / 1 (\text{V/cm}) \\ &= \frac{\text{Current density in A/cm}^2}{\text{Field strength in V/cm}}.\end{aligned}$$

To show that $1 \text{C}^2 \text{cm}^{-3} \text{g}^{-1} \text{s} = 10^7 \Omega^{-1} \text{cm}^{-1}$ we note first that

$$1 \text{C} = 1 \text{A s}, \quad 1 \text{g cm}^2 \text{s}^{-2} = 1 \text{erg} = 10^{-7} \text{J} = 10^{-7} \text{A V s},$$

$$1 \text{A/V} = 1 \Omega^{-1}$$

With these

$$\begin{aligned}1 \text{C}^2 \text{cm}^{-3} \text{g}^{-1} \text{s} &= 1 \text{A}^2 \text{cm}^{-1} \text{s} \times 1 (\text{g cm}^2 \text{s}^{-2})^{-1} \\ &= 10^7 \text{A}^2 \text{cm}^{-1} \text{s} / (\text{A V s}) = 10^7 (\text{A/V}) \text{cm}^{-1} = 10^7 \Omega^{-1} \text{cm}^{-1}.\end{aligned}$$

The measured value of the electrical conductivity of copper is about $5.9 \times 10^5 \Omega^{-1} \text{cm}^{-1}$.

14.5) BOLTZMANN EQUATION AND THERMAL CONDUCTIVITY

(a) Our point of departure is (66), the first-order steady-state solution of the Boltzmann equation (64) in the relaxation time approximation, in the absence of any forces causing acceleration, with only a concentration gradient or a temperature gradient (or both) present in the x-direction. We use the form (68) of f_0 . From (5.12a):

$$\mu = \tau \log(n/n_Q) \quad .$$

With this, (68) becomes

$$f_0 = (n/n_Q) \exp(-\varepsilon/\tau) \quad .$$

According to (3.63), $n_Q \propto \tau^{3/2}$; therefore

$$f_0 \propto n\tau^{-3/2} \exp(-\varepsilon/\tau) \quad . \quad (S1)$$

If the particle concentration is constant, f_0 can depend on position only through τ :

$$\frac{\partial f_0}{\partial x} = \frac{\partial f_0}{\partial \tau} \frac{d\tau}{dx} = \left(-\frac{3}{2\tau} + \frac{\varepsilon}{\tau^2} \right) f_0 \frac{d\tau}{dx} \quad .$$

Insertion into (66) yields (125).

(b) Suppose that $f(\varepsilon)$ were normalized such that

$$\int f(\varepsilon) \mathcal{D}(\varepsilon) d\varepsilon = n \quad . \quad (S2)$$

The particles in the energy interval $(\varepsilon, \varepsilon+d\varepsilon)$ then contribute the particle flux density $v_x f(\varepsilon) \mathcal{D}(\varepsilon) d\varepsilon$. Each particle carries the energy ε . The total energy flux density is

$$J_u = \int v_x \varepsilon f(\varepsilon) \mathcal{D}(\varepsilon) d\varepsilon \quad . \quad (S3)$$

The contribution of the f_0 -term in (S3) vanishes; the remainder is easily seen to yield (126).

(c) If we insert $v_x^2 = 2\varepsilon/3M$ in (126) and write the result in the form (27), $J_n = -K(d\tau/dx)$, we obtain the expression

$$K = (2\tau\tau_c/3m) \int (-3/2 + \varepsilon/\tau) (\varepsilon/\tau)^2 f_0(\varepsilon) \mathcal{D}(\varepsilon) d\varepsilon \quad . \quad (S4)$$

From (S1) and (72):

$$f_0(\varepsilon) \mathcal{D}(\varepsilon) d\varepsilon = C (\varepsilon/\tau)^{1/2} d(\varepsilon/\tau) = C u^{1/2} \exp(-u) du \quad ,$$

where $u = \varepsilon/\tau$ and where C depends on neither ε nor τ . If this is inserted into (S1):

$$C \int_0^{\infty} u^{\frac{1}{2}} \exp(-u) du = C\Gamma(3/2) = \frac{1}{2}C\Gamma(1/2) = C\pi^{\frac{1}{2}}/2 .$$

Here $\Gamma(n)$ is the gamma function, as defined in (A.6), and we have used the relations (A.7) and (A.8). To satisfy (S4), we must have $C = 2n/\pi^{\frac{1}{2}}$. If the normalized value for $f_0(\varepsilon) \mathcal{D}(\varepsilon)d\varepsilon$ is inserted into (S3):

$$K = (4\tau\tau_c/3m\sqrt{\pi}) \int_0^{\infty} (u^{7/2} - (3/2)u^{5/2}) \exp(-u) du .$$

The integral has the value

$$\begin{aligned} \Gamma(9/2) - (3/2)\Gamma(7/2) &= (7/2 - 3/2)\Gamma(7/2) \\ &= 2 \times (5/2) \times (3/2) \times (1/2) \times \pi^{\frac{1}{2}} = 15\pi^{\frac{1}{2}}/4 , \end{aligned}$$

again with the help of (A.6) through (A.8). With this: $K = 5\tau\tau_c/m$, as claimed.

14.6) FLOW THROUGH A TUBE

Consider a cylindrical region with radius $r < a$ within the tube, concentric with the tube walls. The pressure force on this region, $\pi r^2 p$, must equal the viscous force exerted by the boundary of the region with the surrounding liquid. With the aid of (32):

$$\pi r^2 p = -\eta \times 2\pi r L \times (dv/dr) ;$$

$$dv/dr = -(p/2\eta L)r ; \tag{S1}$$

$$v = (p/4\eta L)(a^2 - r^2) . \tag{S2}$$

Here, (S2) is the solution of (S1) with the boundary condition $v(a) = 0$.

The volume flow rate through the tube is

$$\begin{aligned}\dot{V} &= 2\pi \int_0^a v r dr = (\pi p / 2\eta L) \int_0^a (a^2 - r^2) r dr \\ &= (\pi p / 2\eta L) (a^4 / 4) = (\pi a^4 / 8\eta L) p \quad .\end{aligned}\quad (127)$$

14.7) SPEED OF A TUBE

The series combination of a hole and a tube satisfies the same inverse speed addition law (113) as for a pump and a tube:

$$1/S_T = 1/S_{\text{hole}} + 1/S_{\text{tube}} = (1/S_{\text{hole}})(1 + S_{\text{hole}}/S_{\text{tube}}).$$

Inserting (98) and (111), and setting $A = \pi d^2/4$:

$$1/S_T = (4/A\bar{c})(1+3L/4d) = (3/Ad\bar{c})(L+4d/3) \quad ,$$

$$S_T = \frac{Ad\bar{c}}{3} \frac{1}{L+4d/3} = \frac{\pi\bar{c}}{12} \frac{d^3}{L+4d/3} \quad .\quad (S1)$$

The mean speeds for N_2 and O_2 at 273 K = 0°C are given in Table 14.1. Taking $0.8 \bar{c}(N_2) + 0.2 \bar{c}(O_2)$, according to the composition of air, and multiplying by $(293/273)^{1/2}$, to correct for the change from 0°C to 20°C, yields $\bar{c} = 4.60 \times 10^4$ cm/s, and $\pi\bar{c}/12 = 12.04 \times 10^3$ cm/s. Insertion into (S1) expresses the speed in cm³/sec; division by 10³ leads to liter/s.