Phys404 PS7 #7. 2d Fermi gas TA: Jen-Chien Chang

a) For 2d Fermi gas, in analogy to the derivation in page 185,

$\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{\pi n_F}{L}\right)^2$, n_F is the radius of a circle in phase space of n_x, n_y

$\Rightarrow N = 2 \times \frac{1}{4} \times \pi n_F^2 \Rightarrow n_F = \sqrt{\frac{2N}{\pi}}$

$\Rightarrow \varepsilon_F = \frac{\hbar^2}{2m}\pi^2 \frac{2N}{\pi}\cdot\frac{1}{A} = \boxed{\frac{\pi N \hbar^2}{mA}}$

b) $U_0 = 2\times\frac{1}{4}\times 2\pi\int_0^{n_F} n\,dn\cdot\varepsilon_n = \pi\int_0^{n_F}\frac{\hbar^2}{2m}\frac{\pi^2 n^3}{A}dn = \frac{\hbar^2\pi^3}{2mA}\cdot\frac{1}{4}n_F^4 = \frac{\hbar^2\pi^3}{2mA}\cdot\frac{1}{\cancel{4}}\cdot\frac{\cancel{4}N^2}{\pi^2}$

$= \frac{\pi\hbar^2 N}{mA}\cdot\frac{N}{2} = \frac{N}{2}\varepsilon_F$

$\therefore \boxed{\frac{U_0}{N} = \frac{1}{2}\varepsilon_F}$ ... average energy of each particle in ground state

c) from a),

$N(\varepsilon) = \frac{mA}{\hbar^2\pi}\cdot\varepsilon$,

$\Rightarrow \mathcal{D}(\varepsilon) = \frac{dN}{d\varepsilon} = \boxed{\frac{mA}{\hbar^2\pi}} = \frac{N}{E_F}$

(d) for given N, we can determine μ by

$N = \int_0^\infty d\varepsilon\,\mathcal{D}(\varepsilon) f(\varepsilon,\tau,\mu) = \int_0^\infty \frac{\frac{mA}{\hbar^2\pi}d\varepsilon}{e^{\frac{\varepsilon-\mu}{\tau}}+1} = \frac{N}{E_F}\int_0^\infty\frac{e^{\frac{\mu-\varepsilon}{\tau}}d\varepsilon}{1+e^{\frac{\mu-\varepsilon}{\tau}}} = \frac{N}{E_F}\int_0^\infty -\tau\frac{d}{d\varepsilon}\left[\ln\left(1+e^{\frac{\mu-\varepsilon}{\tau}}\right)\right]d\varepsilon$

$= -\frac{N\tau}{E_F}\left[\ln\left(1+e^{\frac{\mu-\varepsilon}{\tau}}\right)\Big|_0^\infty\right] = -\frac{N\tau}{E_F}\left[0-\ln\left(1+e^{\frac{\mu}{\tau}}\right)\right] = \frac{N\tau}{E_F}\ln\left(1+e^{\frac{\mu}{\tau}}\right)$

$\Rightarrow \boxed{\mu = \tau\ln\left(e^{\frac{E_F}{\tau}}-1\right)}$

when $\tau \gg E_F$, $e^{E_F/\tau} = 1+\frac{E_F}{\tau}+\frac{1}{2}\left(\frac{E_F}{\tau}\right)^2+\cdots \approx 1+\frac{E_F}{\tau}$, $\Rightarrow \underline{\mu \approx \tau\ln\frac{E_F}{\tau}}$

when $\tau \ll E_F$, $e^{\frac{E_F}{\tau}}-1 \approx e^{E_F/\tau} \Rightarrow \mu \approx \tau\ln(e^{E_F/\tau}) \approx \underline{E_F}$