Phys 404 Second Miterm Exam Solution

1. Canonical partition function for $\epsilon = cp$ classical rel. particles

a) $$Z(V,T,N) = \frac{1}{N!}\frac{V^N}{h^{3N}}\Bigg[\underbrace{\iiint_{-\infty}^{+\infty} e^{-\epsilon/kT}\,dp_x\,dp_y\,dp_z}_{4\pi\int_0^\infty e^{-cp/kT}p^2dp \;=\; 8\pi\left(\frac{kT}{c}\right)^3}\Bigg]^N$$

$\because \int e^{-p}p^n dp = n!$ $\left(\begin{smallmatrix} n=2 \\ \text{here}\end{smallmatrix}\right)$

$$\therefore Z(V,T,N) = \frac{1}{N!}\left[V\left(\frac{kT}{hc}\right)^3\right]^N \qquad (10\text{ pts})$$

$\ln N! = N\ln N - N$

$$F = -kT\ln Z = -kT\left\{N\ln\left[V\left(\frac{kT}{hc}\right)^3\right] - (N\ln N - N)\right\}$$

Helmholtz energy

$$= -NkT\left\{\ln\left[\frac{V}{N}\left(\frac{kT}{hc}\right)^3\right] + 1\right\}$$

b) Pressure $$p = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = \frac{NkT}{V} \quad \text{or} \quad PV = NkT$$ (5pts)

c) Energy $$U = \bar{E} = -\frac{\partial \ln Z}{\partial \beta} = -T^2\frac{\partial(F/T)}{\partial T} = 3NkT = 3PV$$

$$\therefore u = 3p \qquad (5\text{pts})$$

d) Heat capacity $$C_V = \left(\frac{\partial U}{\partial T}\right)_{V,N} = 3Nk = 3R$$ (5pts)

$$C_p = C_V + R = 4R$$

$$\therefore \gamma = C_P/C_V = \frac{4}{3}$$

2.

a) Debye Temperature : the temperature corresponds to the mode of vibration with highest frequency

for each n, $L = n\frac{\lambda}{2} = \frac{n v_s}{2\nu}$ $\Rightarrow$ $n = \frac{2L}{v_s}\nu$ $\quad (n^2 = n_x^2 + n_y^2 + n_z^2)$

$\therefore$ # of states between n and $n+dn$ is $\overset{\text{3 polarizations}}{3} \cdot \frac{1}{8} \cdot 4\pi n^2 dn$

$\therefore$ $D(\nu)\,d\nu = \frac{3}{8} \cdot 4\pi \left(\frac{2L}{v_s}\nu\right)^2 \frac{2L}{v_s} d\nu = \underline{\underline{\frac{12\pi V}{v_s^3}\nu^2 d\nu}}$ $\quad (V \equiv L^3)$

USE $\int_0^{\overset{\text{Debye frequency}}{\nu_D}} D(\nu)\,d\nu = 3N$ $\Rightarrow$ $\frac{12\pi V}{v_s^3}\frac{\nu_D^3}{3} = 3N$

$\therefore$ $\nu_D = \left(\frac{3N}{4\pi V}\right)^{1/3} v_s = \left(\frac{3n}{4\pi}\right)^{1/3} v_s$ $\quad n \equiv \frac{N}{V}$ atom # density, not the quantum # n above

$\Rightarrow$ $T_D = \frac{h\nu_D}{k_B} = \boxed{\frac{h v_s}{k_B}\left(\frac{3n}{4\pi}\right)^{1/3}}$

b)

$\frac{\text{total \# of excited modes}}{\text{total \# of modes}} = \left(\frac{k_T}{k_D}\right)^3 = \left(\frac{T}{T_D}\right)^3$ $\quad$ radii in k-space

at low temperature,

Energy $E \sim$ (# of excited modes) $\cdot\, k_B T = 3N\left(\frac{T}{T_D}\right)^3 k_B T \sim T^4$

$\therefore$ $C_V = \frac{\partial E}{\partial T} \sim T^3$

3.

a,b) See KK problem 7.2 , $\varepsilon_F = \hbar\pi c\left(\frac{3n}{\pi}\right)^{1/3} = \underline{\hbar\pi c\left(\frac{3N}{\pi V}\right)^{1/3}}$

$$\underline{U_0 = \frac{3}{4}N\varepsilon_F}$$

c) see KK problem 7.10 , $\underline{N = \left(\frac{5}{4}\left(\frac{9\pi}{4}\right)^{1/3}\frac{\hbar c}{GM_H^2}\right)^{\frac{3}{2}}}$

4.

for bosons , $0 \le z \le 1$, $g_{3/2}(z) \le g_{3/2}(1) = \zeta(3/2) \sim 2.612$

$$\frac{N}{V} = \frac{1}{v} = \underbrace{\frac{g_{3/2}(z)}{\lambda^3_{tdB}}}_{\text{excited states}} + \underbrace{\frac{1}{V}\left(\frac{z}{1-z}\right)}_{\text{ground states}}$$

# of particles in excited states $\frac{N_e}{V} = \frac{g_{3/2}(z)}{\lambda^3_{tdB}} \le \frac{g_{3/2}(1)}{\lambda^3_{tdB}}$

The condition for the onset of Bose-Einstein Condensation is

$$\frac{N}{V} > \left(\frac{N_e}{V}\right)_{max}$$

$$\Rightarrow \frac{1}{v_c} = \frac{g_{3/2}(1)}{\lambda^3_{tdB}} \Rightarrow \boxed{v_c = \frac{\lambda^3_{tdB}}{g_{3/2}(1)}} \quad \cdots (b)$$

Recall $\lambda_{tdB} = \frac{h}{\sqrt{2\pi mkT}}$

$$\Rightarrow \quad v_c = \frac{1}{g_{3/2}(1)} \cdot \frac{h^3}{(2\pi mkT_c)^{3/2}}$$

$$\Rightarrow \quad \boxed{T_c = \frac{h^2/2\pi mk}{[v\, g_{3/2}(1)]^{2/3}}} \quad \cdots (a)$$