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Lenz vector and the Coulomb problem

Pauli found the spectrum of Hydrogen using an algebraic method. This method exploits the existence of an extra conserved quantity in the Coulomb problem: the Laplace-Runge-Lenz vector. Classically, this vector points from a focus to the perihelion along the major axis of the orbital ellipse. Its existence is thus contingent on the fact that the orbits in the Coulomb potential are closed. (See, e.g., H. Goldstein, *Classical Mechanics*.) Quantum mechanically, we define the Laplace-Runge-Lenz vector operator:

$$\mathbf{A} = \frac{1}{2m}(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) - \frac{e^2}{r}\mathbf{r}. \tag{1}$$

Since $\mathbf{A}$ commutes with the Hamiltonian H, the operator

$$\mathbf{K} = \sqrt{\frac{-m}{2H}}\,\mathbf{A} \tag{2}$$

is well-defined and Hermitian on bound states (which have negative energy). The following identities hold:

1. $\mathbf{A}$ is hermitian.
2. $[\mathbf{A}, H] = 0$.
3. $\mathbf{K} \cdot \mathbf{L} = 0 = \mathbf{L} \cdot \mathbf{K}$
4. $H = -me^4/2(\mathbf{K}^2 + \mathbf{L}^2 + \hbar^2)$
5. $[K^i, K^j] = i\hbar\epsilon^{ijk}L^k$

Defining $\mathbf{M} = (\mathbf{L} + \mathbf{K})/2$ and $\mathbf{N} = (\mathbf{L} - \mathbf{K})/2$, it follows from 3 that $\mathbf{K}^2 + \mathbf{L}^2 = 2(\mathbf{M}^2 + \mathbf{N}^2)$, and that $\mathbf{M}^2 = \mathbf{N}^2$. Also, it follows from 5, together with the angular momentum commutation relations and the fact that $\mathbf{K}$ is a vector operator, that $[M^i, N^j] = 0$ and the components of $\mathbf{M}$ and $\mathbf{N}$ each separately satisfy the angular momentum commutation relations.

Invoking the general result for the spectrum of the square of an angular momentum operator then yields the Hydrogen spectrum. The eigenvalues of $\mathbf{M}^2$ are $\hbar^2\mathcal{M}(\mathcal{M}+1)$, where $\mathcal{M}$ is an integer or half-integer, and $\mathbf{M}^2 = \mathbf{N}^2$, so 4 yields the spectrum

$$E_n = -\frac{me^4}{2\hbar^2}\frac{1}{4\mathcal{M}(\mathcal{M}+1)+1} = -\frac{me^4}{2\hbar^2}\frac{1}{n^2}$$

where the principal quantum number $n = 2\mathcal{M}+1$ has been introduced. Note that although $\mathcal{M}$ can be a half-integer, n is always an integer.

The degeneracy of the level E_n is the product of the degeneracies $2\mathcal{M} + 1$ for the eigenstates of $\mathbf{M}^2$ and $\mathbf{N}^2$ with eigenvalue $\hbar^2\mathcal{M}(\mathcal{M} + 1)$, so the overall degeneracy is $(2\mathcal{M} + 1)^2 = n^2$. Note that eigenstates of M_z and N_z are eigenstates of $L_z = M_z + N_z$, but they are *not* eigenstates of $\mathbf{L}^2$.

The existence of the Lenz vector, an "extra" operator that commutes with the hamiltonian, explains the "accidental" degeneracy of the Coulomb spectrum. If $|E\rangle$ is an eigenstate of H with eigenvalue E, then so is $A^i|E\rangle$, where A^i is any component of the Lenz vector. Thus $A^i|E\rangle$ must be some linear combination of the states with energy E. Since A^i does not commute with L^2, however, $A^i|E\rangle$ will not have the same total angular momentum as did $|E\rangle$. Thus there are states with different angular momentum but the same energy. This is the origin of the accidental degeneracy. In fact the degeneracy of the Coulomb spectrum is maximal: every angular momentum less than the principal quantum number occurs in a given energy level. We can see this by acting with the combination $A_+ = A_x + iA_y$ on the top state $|Ell\rangle$ for a given l. It is easy to show that $A_+|Ell\rangle = c_l|E, l + 1, l + 1\rangle$ for some constant c_l. Thus, as long as c_l is not zero, we can continue to raise the angular momentum. We know that it cannot be raised beyond one less than the principal quantum number N, since the maximum possible l for the level N is $N - 1$, so evidently $c_{N-1} = 0$. Although we have not shown it explicitly, c_l is not zero until l reaches $N - 1$.

What about the unbound states? For these H is *positive*, so $\mathbf{K}$ as defined in (2) is *anti*-hermitian, so $\mathbf{M}$ and $\mathbf{N}$ are a combination of hermitian and anti-hermitian operators. Thus even though they satisfy the angular momentum commutation relations, we can not use the standard results for the spectrum of angular momentum operators.