Topic 6 The Rayleigh-Taylor Mode

[A stable/unstable mode of an inhomogeneous equilibrium]

Waves in inhomogeneous media

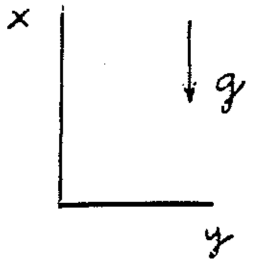
In the preceding topic, we treated small amplitude waves of a homogeneous collisional gas. Since the equilibrium was not dependent on any spatial coordinate, we found we could have plane wave solutions of the form

$$\exp[i\vec{k} \cdot \vec{x} - i\omega t] \; . \tag{1}$$

In general, if we have an inhomogeneous equilibrium, it is not possible to assume a plane wave solution since the resulting PDE obtained from the linearized equations has non-constant coefficients. For example, if $\vec{g} \neq 0$, then equilibrium under gravity demands

$$\vec{\nabla} p = n m \vec{g}$$

$$\text{or } \frac{\partial p}{\partial x} = -nmg \, ,$$



$$p = p_0(x) \; , \; n = n_0(x), \; T = T_0(x).$$

In this case, we may not Fourier analyse in the x coordinate, although we may still do so in the y and z coordinate. The most general perturbation may then be written

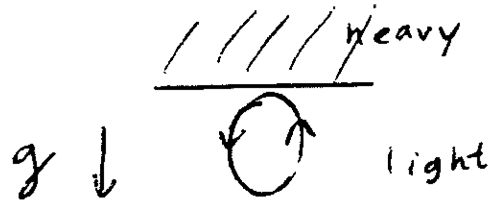
$$\hat{n} = \hat{n}(x) \exp (ik_y y - i\omega t), \text{ etc.} \qquad (2)$$

where k_z has been set to zero without loss of generality.

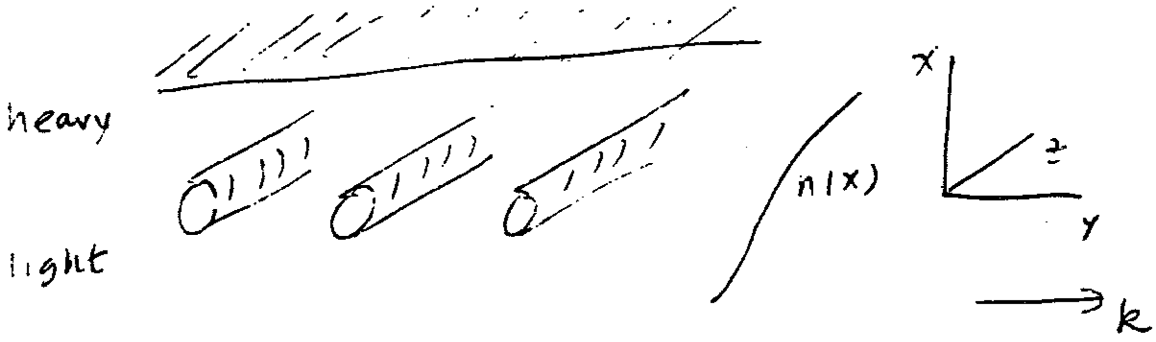
In general, each of the five modes of the homogeneous case now carry over to the inhomogeneous case with each mode possibly being modified by a term proportional to g. Thus, to investigate all the modes of our inhomogeneous equilibrium, we should solve our linearized equations all over again but this time with the dependencies (2), and the new equilibrium condition $\partial p_o/\partial x = -n_o mg$.

However, the effect of g is not dramatic for all the five modes. In particular, for practical parameters, g is "small" for sound waves and its effect on these waves is negligible (from everyday experience, gravity does not impair our hearing it seems). In addition, physical intuition may lead us to notice that the effect of g on a convective cell which swirls <u>in the y-z plane</u> is bound to be negligible (i.e. the mode with $\vec{k} = \vec{k}_y$, $\hat{u}_z \neq 0$). (see picture below)

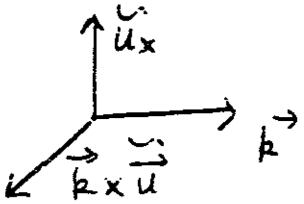
However, a convective cell set up in the x-y plane is likely to be influenced by g since we are mixing matter around ($\partial n_o/\partial x \neq 0$) and thus changing the gravitational potential energy of the system.
(see picture)



Likewise, if we create a density spike with a corresponding temperature hole (entropy mode), then this mode is also likely to be influenced by gravity because creating a density spike corresponds to trying to float a heavier rod in a lighter fluid.



Thus, in this section, we shall be primarily interested in the effect of $\vec{g}$ on the entropy mode and on the convective cell with $\tilde{u}_x \neq 0$ (or, more precisely, the convective cell with $\vec{k} \times \vec{\tilde{u}} \propto \vec{\nabla} \times \vec{\tilde{u}}$ in the $\hat{z}$ direction). The modified modes which result due to the presence of $\vec{g}$ are coupled and they are called Rayleigh-Taylor modes. These modes may be unstable; hence the interest in these modes.



In deriving the equations governing the RT mode, we shall start with the linearized equations again but this time include g and $n_o = n_o(x)$ with a dependence of the form (2). We shall also make use of the physical facts we have just intuited - namely,

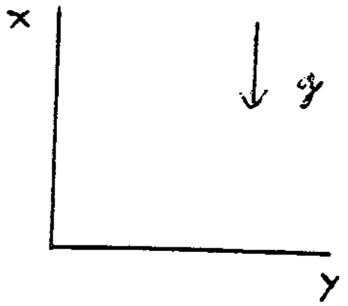
(i) the convective cell is in the x-y plane and $\hat{z} \cdot (\hat{\vec{\nabla}} \times \hat{\vec{u}}) \neq 0$,

(ii) $\hat{p} \approx 0$ for both the homogeneous convective cell and the entropy mode.

Fact (ii) was true in the homogeneous case. There is no real reason why the eigenvector in the inhomogeneous case should also have $\hat{p} \approx 0$. But, physical intuition tells us that since the modes we are examining are not <u>essentially</u> dynamic modes, $\hat{p} \approx 0$ is a possibility. In any case, we shall self-consistently check any *a priori* assumptions.

The eigenmode equation for RT modes

(a) Equilibrium



$$\frac{\partial p_o}{\partial x} = - n_o mg$$

$$n_o = n_o(x); \; T_o = T_o(x)$$

$$p_o \equiv n_o T_o \, , \quad \partial n_o/\partial x \neq 0 \, . \text{ in general} \qquad (3)$$

(b) Linearized equations

$$\frac{\partial \tilde{n}}{\partial t} + \tilde{u}_x n_o' + n_o \vec{\nabla} \cdot \vec{\tilde{u}} = 0. \tag{4a}$$

$$n_o \frac{m \partial \vec{\tilde{u}}}{\partial t} = - \vec{\nabla} \tilde{p} + \tilde{n} m \vec{g} , \tag{4b}$$

$$\frac{\partial}{\partial t} (\tilde{p} - \gamma T_o \tilde{n}) + \underset{\wedge}{\overset{P_o}{}} \tilde{u}_x S_o' = 0 , \tag{4c}$$

where $S_o' \equiv (d/dx) \ln (p/n^{\gamma})$. We notice, in the linearized equations, the presence of additional terms proportional to n_o', S_o' and g. These are from the inhomogeneity and gravity.

To make life simpler, we now <u>use</u> the fact that the main motion is a $\vec{\nabla} \times \vec{u}$ which is in the $\hat{z}$ direction. Therefore, we take the curl of (4b) and then dot it with $\hat{z}$. We get,

$$m \hat{z} \cdot (\vec{\nabla} \times n_o \frac{\partial \vec{\tilde{u}}}{\partial t}) = m\hat{z} \cdot (\vec{\nabla} \times \tilde{n}\vec{g}),$$

where we have used $\vec{\nabla} \times \vec{\nabla} \tilde{p} \equiv 0$. Now using $\vec{\nabla} \times n_o \vec{\tilde{u}} \equiv \vec{\nabla} n_o \times \vec{\tilde{u}} + n_o \vec{\nabla} \times \vec{\tilde{u}}$, and $\vec{\nabla} \times \vec{g} = 0$, we obtain

$$\hat{z} \cdot [n_o \vec{\nabla} \times \frac{\partial \vec{\tilde{u}}}{\partial t} + \vec{\nabla} n_o \times \frac{\partial \vec{\tilde{u}}}{\partial t}] = \hat{z} \cdot (\vec{\nabla} \tilde{n} \times \vec{g}). \tag{4d}$$

Finally, writing out in components, we get

$$\frac{\partial}{\partial t} [n_o (\tilde{u}_y' - \partial \tilde{u}_x / \partial y) + n_o' \tilde{u}_y] = g(\partial \tilde{n} / \partial y).$$

To make like $^{\text{life}}$ even more simple, we now use the fact that $\tilde{p} \approx 0$, or we surmise this to be the case. $\tilde{p}$ only appears in (4c). So, $\tilde{p} \approx 0$ really corresponds to making the <u>a priori</u> assumption that the term proportional to $\hat{p}$ in (4c) is small compared to any of the other terms in that equation. Thus, we assume, to be shown self consistenly later on, that

$$\hat{p} \ll \gamma\, T_o \hat{n} \qquad (4d')$$

in which case, (4c) reduces to

$$\gamma(\partial/\partial t)(\hat{n}/n_o) + \hat{u}_x\, S_o' = 0. \qquad (4e)$$

We may now verify that (4a), (4d), and (4e) constitute a system of 3 equations in the 3 unknowns $\hat{n}$, $\hat{u}_x$, and $\hat{u}_y$.

(c) Fourier Analysis

We now let

$$\hat{n} = \hat{n}\,(x)\,\exp\,(ik_y y - i\omega t), \text{ etc.} \qquad (5)$$

In this case, $\partial/\partial t \rightarrow -\,i\omega$, $\partial/\partial y \rightarrow ik_y$, and (4a), (4d), and (4e) reduce to:

$$-i\omega\hat{n} + \hat{u}_x\, n_o' + n_o \hat{u}_x' + n_o\; ik_y\; \hat{u}_y = 0 \qquad (6a)$$

$$-\,i\omega[n_o(\hat{u}_y' - ik_y \hat{u}_x) + n_o'\; \hat{u}_y] = ik_y\; g\hat{n} \qquad (6b)$$

$$i\omega\gamma(\hat{n}/n_o) + \hat{u}_x\; S_o' = 0. \qquad (6c)$$

Adding (6a) to (n_o/γ) times (6c), we get

$$\hat{u}_x(n_o/\gamma)\, S_o' + \hat{u}_x\, n_o' + n_o\hat{u}_x' + ik_y\, n_o\hat{u}_y = 0$$

$$\Rightarrow \qquad \left(\hat{u}_x' + \frac{p_o'}{\gamma p_o}\hat{u}_x\right) = -ik_y\, \hat{u}_y\ . \tag{7}$$

We now use the expression (7) for $\hat{u}_y$ in (6b), and expression (6c) for $\hat{n}$ in the RHS of (6b). This gives us the eigenvalue equation for $\hat{u}_x$:

$$[n_o(\hat{u}_x' + \frac{1}{\gamma}(\ell n p_o)'\ \hat{u}_x)]' - k_y^2 n_o\ \hat{u}_x = -(\frac{k_y}{\omega})^2\ \frac{gS_o'}{\gamma}\ \ n_o\ \hat{u}_x\ .$$

To have any hope of solving this equation, we make the further assumptions

$$\frac{u'}{u} \gg \frac{n_o'}{n_o}, \frac{p_o'}{p_o}\quad , \ u'' \gg u'\ \frac{p_o'}{p_o}\quad , \tag{8}$$

i.e., the perturbed structures are sharper than equilibrium scales. With this assumption, we have, finally

$$u'' - k_y^2\ u \simeq -k_y^2\ \ (\frac{gS_o'}{\gamma\omega^2}\ )\ u \tag{9}$$

where $u \equiv \hat{u}_x$, $S_o' = S_o'(x)$, with assumptions (8) and (4d′). Before solving (9), we note the following:

(i) if $u''/u \ll k^2_y$, i.e., the structuring in y is sharper than that in x (long, thin cells), then we may throw away the u'' term and get

$$\omega^2 \simeq g\ S_o'/\gamma\ .$$

Thus, the frequency of oscillation is real if $S_o' > 0$ and imaginary (instability) if $S_o' < 0$;

(ii) $S_o' \equiv (p_o'/p_o - \gamma n_o'/n_o)$. If the scale size of n_o is very short compared to the scale size for p_o (and, for the atmosphere, the scale size of p_o from $dp_o/dx = - n_o mg$ may be estimated to be $\sim$ 5 Km.), then $S_o' \approx - \gamma n_o'/n_o$. In that case,

$$\omega^2 \approx - g(n_o'/n_o) ,$$

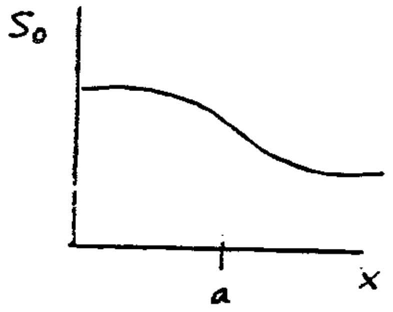
and stability is determined by dn_o/dx with instability if the density increases as we go up.

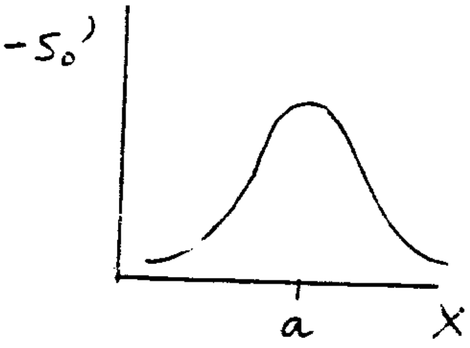
The full eigenvalue equation (9) must be solved, however, to get a mathematically satisfying result for ω. (ω cannot be a function of x as we have obtained.)

Solution of the eigenvalue equation (9)

We have assumed that $\tilde{u}'/\tilde{u} >> n_o'/n_o$, etc. So we must have solutions which are localized, or have structures, on scales much shorter than n_o/n_o'. Since our "local" analysis earlier gave us $\omega^2 \propto S_o'$, we expect the convection to be maximized where S_o' is a maximum.

Consider the following:





Suppose S_o' peaks at a. We then assume that the convection will localize itself in a region about a and expand $S_o'(x)$ about a, viz., $S_o'(x) \approx S_o'(a) + S_o'''(x-a)^2/2$. We next shift our coordinate system from x to x − a. Thus if we let $s \equiv x - a$, we may rewrite (9) as

$$u_{ss} - k_y^2 \left[(1 + \gamma_g{}^2/\omega^2) - \left(\frac{\gamma_g{}^2}{\omega^2}\right)\frac{s^2}{L^2}\right] u = 0, \tag{14}$$

B.C. $u \to 0$ as $s \to \pm\infty$,

where $\gamma_g^2 \equiv -gS_o'/\gamma$, and $L^{-2} \equiv -(S_o'''/S_o')/2$.

This is just the familiar Harmonic oscillator equation. It is well known that for

$$y'' + \frac{1}{x_o^2}(\alpha - x^2/x_o^2)\, y = 0, \; y \to 0 \text{ at } \pm\infty,$$

$$y_n = y_o H_n(x/x_o), \; y_o = e^{-1/2\,(x/x_o)^2}$$

$$\alpha = \alpha_n = \frac{2n+1}{2}, \qquad \text{Re } x_o{}^2 > 0. \tag{15}$$

Comparing (14) and (15), we obtain the relation

$$(kx_o)^4 = -(kL)^2 (\omega/\gamma_g)^2, \tag{16a}$$

$$\text{and } (1 + \gamma_g{}^2/\omega^2) = -(2n+1)/(kx_o)^2, \tag{16b}$$

$$\mathrm{Re}(x_o^2) > 0.$$

From (16), we have to solve for ω and x_o in terms of k, γ_g, and n. The eigenfunction is

$$u = u_n = e^{-1/2(s/x_o)^2} H_n(s/x_o). \qquad (17)$$

(16) and (17) have to be compatible with our original assumptions (8) and (12) and further we must satisfy Re $x_o{}^2 > 0$. Assumptions (8), (12) and (15) translate in to

$$x_o \ll L \qquad (18)$$

$$\mathrm{Re}\ x_o^2 > 0\ . \qquad (18b)$$

To solve (18a) and (18b), we solve (16) for $(kx_o)^2$. From (16), eliminating ω^2, we have

$$(kx_o)^4 + (2n + 1)(kx_o)^2 - (kL)^2 = 0$$

$$\Longrightarrow \quad 2(kx_o)^2 = -(2n + 1) \pm \sqrt{(2n+1)^2 + 4(kL)^2)}\ .$$

<u>Now</u>, Re $x_o{}^2 > 0$

$\Longrightarrow$ take + sign only for $n = 0,1,2,\ldots\ldots.$

$$\Longrightarrow \quad (kx_o)^2 = \frac{1}{2}\sqrt{(2n+1)^2 + 4(kL)^2} - \frac{(2n+1)}{2}$$

Further when $kL \ll 2n + 1$

$$(kx_o)^2 \simeq (kL)^2$$

$\Longrightarrow x_o{}^2 \simeq L^2$. This violates (18a)

Thus the only allowable possible solutions are for $kL \gg 2n + 1$.

For $kL \gg 2n + 1$,

$$(kx_o)^2 \simeq (kL) \tag{19}$$

$$\left(\frac{x_o}{L}\right)^2 \simeq \frac{1}{kL} \ll 1, \quad \text{consistent with (18a).}$$

Using (19) in (16b), we get the dispersion relation.

$$\omega^2 = \frac{-\gamma_g^2}{1 + (2n+1)/(kL)} \tag{20}$$

$$\gamma_g^2 \equiv -g\ S_o'/\gamma \qquad\qquad k^{-1} \ll x_o \ll L$$

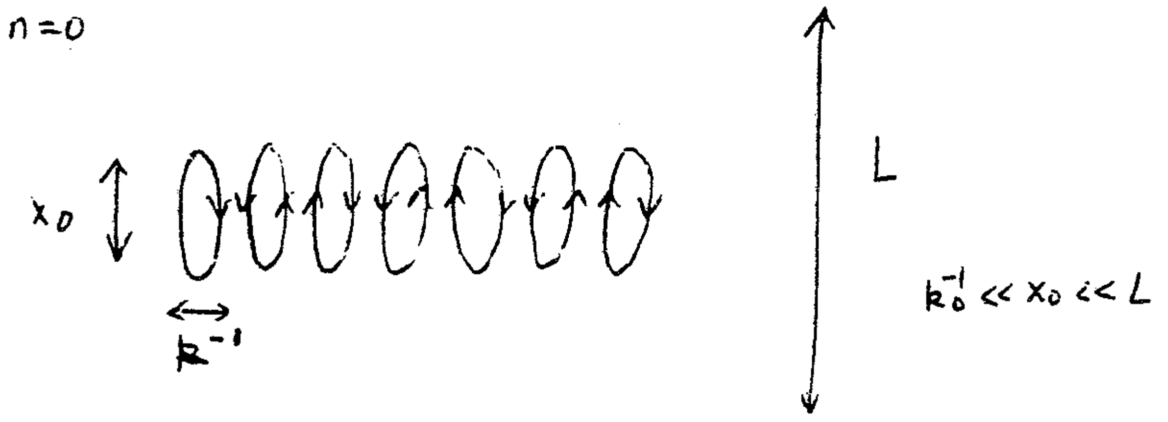
The stability criterion is

$S_o' > 0$

or, if $p'/p \ll n'/n$,

$n_o'/n_o < 0$.

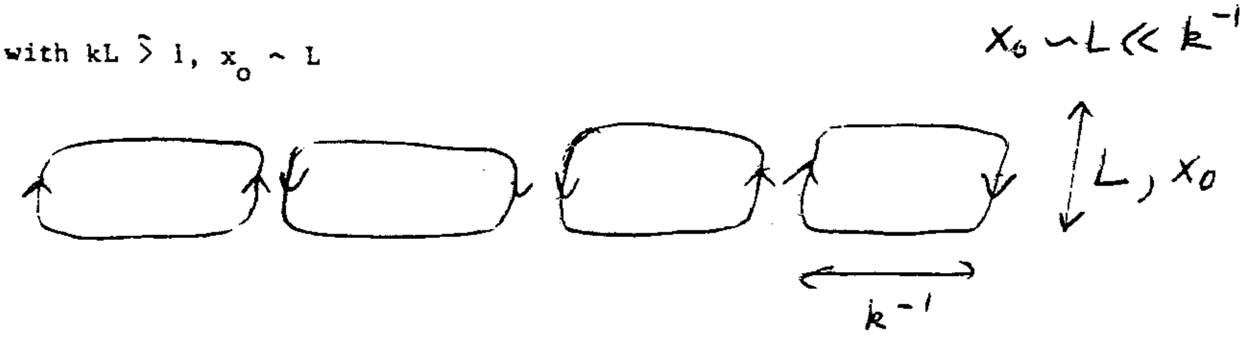
Our modes, for $kL > 1$ and $n = 0$, look like



If we had been able to relax the assumption $x_o < L$, i.e., allowed $u'/u \sim 1/L$, then our full dispersion might have been

$$\omega^2 = \frac{-\gamma_g^{\,2}}{1 + (2n+1)(kx_o)^{-2}}$$

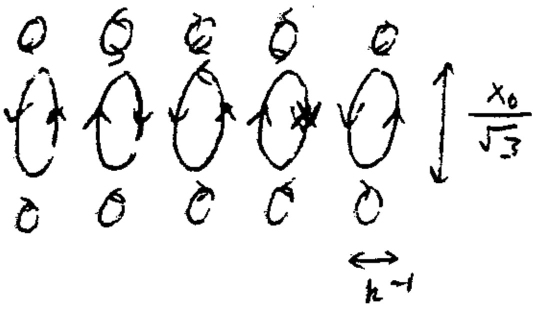
with $kL \gtrsim 1$, $x_o \sim L$



Note: if $(kx_o)^2 \ll 2n + 1$

$$\omega^2 \approx -\gamma_g^{\,2} \frac{(kx_o)^2}{2n+1} < \gamma_g^{\,2} \quad .$$

Thus the maximum growth rate (in sweeping k's) is for modes with $kL \gg 1$ and $n = 0$ [see (20)]. This rate is $\omega^2 = -\gamma_g^{\,2}$. For $kL > 1$ and $n = 3$, the modes look like



Is $\hat{p} \approx 0$? (self consistency check.)

We recall that, early in (Eq. (4d´)), we had assumed

$$\frac{\hat{p}}{p_o} << \hat{n}/n_o. \qquad (4d')$$

Now that we have solved the problem, we must check this assumption for self consistency. To do this, we first search for an equation for $\hat{p}$. Consider the $\hat{y}$ component of (4b). This gives:

$$nm \frac{\partial}{\partial t} \hat{u}_y = - ik_y \hat{p}$$

$$\Longrightarrow \qquad \hat{p} \sim \frac{nm\omega}{k} \hat{u}_y .$$

But from (7),

$$\hat{u}_y \sim \frac{\hat{u}_x'}{k}$$

$$\therefore \qquad \hat{p} \sim \frac{nm\omega}{k^2} \hat{u}_x' \sim \frac{nm\omega}{k^2} \frac{\hat{u}_x}{x_o} .$$

Again from (6c),

$$\frac{\hat{n}}{n} \sim \frac{\hat{u}_x}{\omega} \frac{n'}{n}$$

$$\Longrightarrow \quad \hat{p} \sim \frac{nm\omega^2}{k^2 n'} \frac{\hat{n}}{x_o} . \qquad (22)$$

Using (22) in (4d′), we get the condition

$$\omega^2 << (kc_s)^2 \, (x_o/L)$$

which reduces, upon using (19) and (20), to

$$g << (kc_s)^2 \, (L/k)^{1/2} .$$

This is easily satisfied.

Also, we need $\partial_t \gg \vec{\breve{u}} \cdot \vec{\nabla}$. But $\partial_t \sim \sqrt{g/L}$.

$\Rightarrow \quad g/L \gg k^2 \breve{u}^2$

$\Rightarrow \quad \breve{u}^2 \ll \frac{g}{k} \frac{1}{kL}$

$\Rightarrow \quad \rho \breve{u}^2 \ll \rho g L / (kL)^2$

$\Rightarrow$ Kinetic energy $\ll$ potential energy.