Topic 4

Some physics of the fluid equations

The ideal fluid equations are:

\[ \frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{u}) = 0 \]  \hspace{1cm} (1) \hspace{1cm} \text{(conservation of mass)}

\[ n \frac{d \mathbf{u}}{dt} = - \mathbf{\nabla} p + n \mathbf{f} \]  \hspace{1cm} (2) \hspace{1cm} \text{($F = ma$)}

\[ \frac{d}{dt} \left( \frac{p}{n \gamma} \right) = 0 \]  \hspace{1cm} (3) \hspace{1cm} \text{($p \gamma = \text{const.}$)}

We will discuss several aspects of the fluid equations in what follows.

1. Equilibrium

If a gas is in hydrostatic equilibrium, then \( u = 0 \) and \( \frac{d}{dt} = 0 \). In this case, the forces on the RHS of (2) must balance, i.e.

\[ \dot{\mathbf{f}}_p = n \mathbf{f} \]

Suppose \( \dot{\mathbf{f}} = 0 \).

Then \( \dot{\mathbf{v}}_p = 0 \Rightarrow \) there must be no pressure variations in space. However, since \( p = nT \), we may have the following situation (as in the outside of a building in winter as opposed to the inside):

\[ n(x) \quad T(x) \]

\[ x \]
i.e., $T \frac{\partial n}{\partial x} = -n \frac{\partial T}{\partial x}$.

Of course, this corresponds to a localized hot spot and the above equilibrium will decay; the decay will, however, be on a slow time scale. What do we mean by slow time scale? We mean slow compared to the hydrodynamic time scale. The hydrodynamic time scale is the time scale over which imbalances in the RHS of (2) are wiped out. So, for instance, if there exists a pressure difference between the outside and inside of buildings, then that difference is wiped out by a blast of air on a time scale given by

$$nm \frac{du}{dt} \sim \nabla p \sim \frac{P}{L} \frac{\Delta p}{p}$$  \hspace{1cm} (4)

where $\Delta p/p$ is the fractional pressure difference.

If $\Delta T = 30^\circ C$ and $\Delta n = 0$, then $\Delta p/p \sim 1/10$.

Then, from (4), with $\frac{du}{dt} \sim \frac{u}{t} \sim \frac{L}{t}$,

we have

$$\tau_H \sim \sqrt{p/\Delta p} \left( \frac{L}{c_s} \right).$$

For $L \sim 3m$, $c_s \sim 300 \text{ m/s}$, we have $\tau \sim 30 \text{ milliseconds}$.

Compare this to the time scale for heat diffusion which is $\tau_D \sim \frac{L^2}{\lambda^2 v} \sim \text{ hours}$ (see appendix $\gamma$).

In fact, $\frac{\tau_H}{\tau_D} \sim \frac{L}{c_s \frac{\lambda}{L}} \ll 1$. 
2. **Equilibrium with g**

**hydrostatic**

Consider equilibrium in gravity.

Then \( \ddot{F} = mg \), \( \ddot{g} = -y g \)

\[
\frac{\partial p}{\partial y} = -nmg .
\]

So the pressure profile must be such that pressure decreases as we go up.

Again, because \( nT = p \), there is a degree of freedom in how we choose \( n(y) \) or \( T(y) \). That is, either of the below situations are acceptable equilibria:

(I) is probably the usual condition. Even though (II) is acceptable, an inverted density profile does not stay that way for long because it is an unstable equilibrium. An unstable equilibrium is analogous to the ball-on-hill analogy.
Ball on hill/well analogy

Why is (II) unstable? The answer is that denser fluid is resting on top of lighter fluid. This situation cannot remain so under perturbations.

A more illuminating answer, based on everyday observation, is that "hot air rises". So "convection currents" will be set up to bring the heavier, colder air down below. This interchange of air masses is known as the "Rayleigh-Taylor" or "Interchange" instability. It is a feature in atmospherics, stellar equilibria, and plasma equilibria. We shall deal with this instability in more detail later.

What is the time scale for the RT instability? Since it is driven by gravity, \( nm \frac{du}{dt} \sim nmg \), \( \frac{du}{dt} \sim L/\tau^2 \)

\[
\Rightarrow \quad \tau^2 \sim g/L, \quad g \sim 980 \text{ cm/s}^2.
\]

**Time scale**

If \( L \sim 3m \), then \( \tau \sim 3 \text{ sec} \). This is compatible with experience.

3. **Navier Stokes equations**

We derived the "ideal" fluid equations by simply using \( f = f_0 \) = maxwellian in the moment equations. If we also include \( f_1 \), then we
obtain the so called "Navier Stokes" equations. In general, for $f_1$, 
$\dot{\mathbf{v}} \cdot \dot{\mathbf{p}} = \dot{v}_p$, i.e. the off diagonal elements of $\dot{\mathbf{p}}$ are $\neq 0$. Also, $\dot{q} \neq 0$.

Using $f_1$ in our previous consistency conditions we can obtain the following set of transport equations:

$$\frac{\partial u}{\partial t} + \mathbf{v} \cdot \mathbf{u} = 0$$  \hspace{1cm} (1a)

$$nm \frac{du}{dt} = -\dot{v}_p + n\dot{\mathbf{p}} + nm\mu \left[ \frac{\dot{\mathbf{v}} \cdot \mathbf{u}}{3} + \mathbf{v}^2 \mathbf{u} \right]$$  \hspace{1cm} (2a)

$$\frac{3}{2} n \frac{dT}{dt} + nT \dot{\mathbf{v}} \cdot \mathbf{u} = -\dot{\mathbf{v}} \cdot \mathbf{q}$$  \hspace{1cm} (3a)

$$\dot{q} \equiv -\kappa \dot{\mathbf{v}} T$$

$$\mu \equiv C_u \lambda^2 v$$
$$\kappa \equiv C_k \lambda^2 v$$

where $C_u$, $C_k$ are numerical coefficients of order unity.

These equations are not "ideal" because they include viscosity and heat conduction.

4. Physical meaning of pressure and viscosity

Meaning of pressure (particle point of view)

When particles with momentum $m\mathbf{v}$th embed themselves or escape from a fluid element, that element gains or loses momentum. We show that if there is a pressure differential across an element, then that difference manifests itself in a rate of change of average momentum of the

*assuming, of course, we have solved explicitly for $f_1$.  

element. We also show that this force is proportional to $\dot{v}_p$. Consider the situation below:

\[ \lambda \ll \hat{L} \ll L \]

Let $\dot{v}_n = \partial n / \partial x \neq 0$.

Then in a time $v^{-1}$, particles at the edge a distance $\lambda$ away enter or leave the element. Net moment gain is then

\[ \Delta p_+ = m v_{th} \lambda (n_1 + n_2 - n_3 - n_4) \]

Since $\lambda \ll \hat{L}$, $n_1 = n_2$ and $n_3 = n_4$; also $n_2 = n_3 + |\partial n / \partial x| \hat{L}$ since $\hat{L} \ll L$.

\[ \Delta p_+ = \lambda A m v_{th} |\partial n / \partial x| \hat{L}. \]

Since this gain happens on time $v^{-1}$, the rate of change of momentum is $\lambda A L m v_{th} |\partial n / \partial x| v$.

This must be equated with mass $x$ acceleration.

\[ \therefore \lambda \ A L m v_{th} (\partial n / \partial x) v = n m A L \frac{du}{dt}. \]

Using $\lambda v = v_{th}$ and $v_{th}^2 = T/m$, we have
\[ \text{We have obtained order of magnitude agreement} \]
\[ \text{with } \frac{\partial u}{\partial t} = - \frac{\partial}{\partial x} \]
\[ \frac{2n}{3} \]

**Meaning of viscosity**

Consider the sheared flow situation shown below.

For this flow, \( u_2 \) will drag on \( u_1 \), \( u_1 \) on \( u_0 \), and so on. What is the magnitude of the drag force?

Again, we compute the rate of change of directed momentum of an element in streamline \( u_1 \).

Random motions of particles such as 1 and 2 add momentum to \( u_1 \).

Random motions 3 and 4 take away momentum.

Nett increase in momentum in \( v^{-1} \)

\[ u_0 \quad u_2 \quad u_1 \]
\[ = \quad \text{nm} \left( \frac{v_1}{b} + \frac{v_2}{2} - 2v_4 \right) \]
But \[ u_2 = u_1 + \lambda |u_1'| + \frac{\lambda^2}{2} u_1'' \]

\[ u_0 = u_1 - \lambda |u_1'| + \frac{\lambda^2}{2} u_1'' \]

\[ \Rightarrow \Delta p_+ = nm \lambda^2 u_1'' \]

Mass x accel. = \[ nm \lambda^2 \nu u_1'' \]

\[ \Rightarrow \frac{du_1}{dt} \sim \lambda^2 \nu u_1'' , \Rightarrow \tau \sim \frac{L^2}{\lambda^2 \nu} . \]

Note this agrees with the LHS and RHS (last term) of Eq. (2a).

Note: the above type of accounting may also be applied to understand thermal conduction.

Final Note

Recall that the fluid equations were obtained by an expansion of the Boltzmann equation assuming

\[ \nabla \gg 1, \lambda \ll L. \]

For the time scales obtained by us so far, namely

\[ \tau_H \sim L/c_s , \quad \tau_D \sim \frac{L^2}{\lambda^2 \nu} , \]

we must make sure that (self-consistently)
\[ \nu \tau_H \gg 1, \quad \nu \tau_D \gg 1. \]

This is true since

\[ \nu \tau_H \sim L/\lambda \gg 1, \quad \nu \tau_D \sim (L/\lambda)^2 \gg \nu \tau_H \gg 1. \]