Topic 4 Some physics of the fluid equations

The ideal fluid equations are:

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot n\vec{u} = 0 \qquad (1) \qquad \text{(conservation of mass)}$$

$$nm \frac{d\vec{u}}{dt} = - \vec{\nabla}p + n\vec{F} \qquad (2) \qquad (F = ma)$$

$$\frac{d}{dt} (p/n^{\gamma}) = 0 \qquad (3) \qquad (pV^{\gamma} = \text{const.})$$

We will discuss several aspects of the fluid equations in what follows.

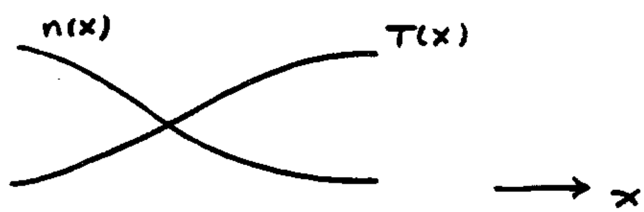
1. Equilibrium

If a gas is in hydrostatic equilibrium, then $\vec{u}=0$ and $\partial/\partial t = 0$. In this case, the forces on the RHS of (2) must balance, i.e.

$$\vec{\nabla}p = n\vec{F} .$$

Suppose $\vec{F} = 0$.

Then $\vec{\nabla}p = 0 \Rightarrow$ there must be no pressure variations in space. However, since $p = nT$, we may have the following situation (as in the outside of a building in winter as opposed to the inside):



i.e., $T\,\partial n/\partial x = -n\partial T/\partial x$.

Of course, this corresponds to a localized hot spot and the above equilibrium will decay; the decay will, however, be on a slow time scale. What do we mean by slow time scale? We mean slow compared to the hydrodynamic time scale. The hydrodynamic time scale is the time scale over which imbalances in the RHS of (2) are wiped out. So, for instance, if there exists a pressure difference between the outside and inside of buildings, then that difference is wiped out by a blast of air on a time scale given by

$$nm\,\frac{du}{dt} \sim \nabla p \sim \frac{p}{L}\left(\frac{\Delta p}{p}\right) \tag{4}$$

where $\Delta p/p$ is the fractional pressure difference. If $\Delta T = 30^{\circ}C$ and $\Delta n = 0$, then $\Delta p/p \sim 1/10$.

Then, from (4), with $\frac{du}{dt} \sim \frac{u}{\tau} \sim \frac{L}{\tau^2}$,

we have

$$\tau_H \sim \sqrt{p/\Delta p}\,(L/c_s).$$

For $L \sim 3m$, $c_s \sim 300$ m/s, we have $\tau \sim 30$ millisecs.

Compare this to the time scale for heat diffusion in gases which is $\tau_D \sim L^2/\lambda^2\nu \sim$ hours (see appendix ~~1~~ 2).

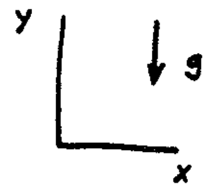
In fact, $\frac{\tau_H}{\tau_D} \sim \frac{L}{c_s}\frac{\kappa}{L^2} \sim \frac{\lambda}{L} \ll 1$.

2. Equilibrium with g

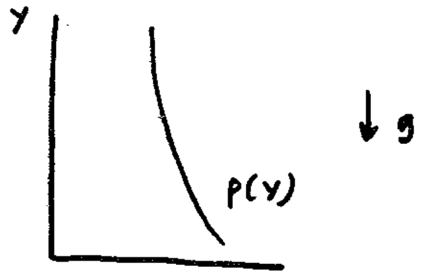
Consider hydrostatic equilibrium in gravity.

Then $\vec{F} = m\vec{g}$, $\vec{g} = -\hat{y}\, g$

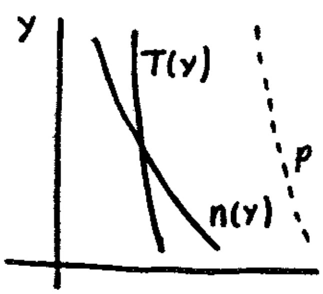
$$\frac{\partial p}{\partial y} = - nmg \; .$$



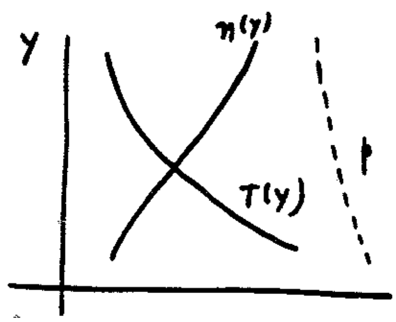
So the pressure profile must be such that pressure decreases as we go up.



Again, because $nT = p$, there is a degree of freedom in how we choose $n(y)$ or $T(y)$. That is, either of the below situations are acceptable equilibria:



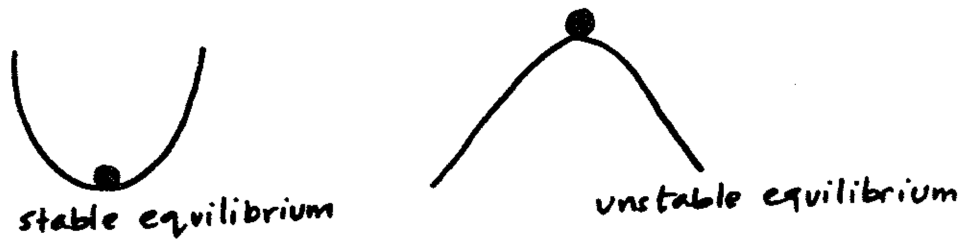
(I)



(II)

(I) is probably the usual condition. Even though (II) is acceptable, an inverted density profile does not stay that way for long because it is an unstable equilibrium. An unstable equilibrium is analogous to the ball-on-hill analogy.

Ball on hill/well analogy



Why is (II) unstable? The answer is that denser fluid is resting on top of lighter fluid. This situation cannot remain so under perturbations.

A more illuminating answer, based on everyday observation, is that "hot air rises". So "convection currents" will be set up to bring the heavier, colder air down below. This interchange of air masses is known as the "Rayleigh-Taylor" or "Interchange" instability. It is a feature in atmospherics, stellar equilibria, and plasma equilibria. We shall deal with this instability in more detail later.

What is the time scale for the RT instability? Since it is driven by gravity, $nm\, du/dt \sim nmg$, $du/dt \sim L/\tau^2$

$$\Rightarrow \quad \tau^2 \sim g/L \;, \qquad g \sim 980 \text{ cm/s}^2.$$

If $L \sim 3$m, then $\tau \sim \sqrt{3}$ sec. This time scale is compatible with experience.

3. Navier Stokes equations

We derived the "ideal" fluid equations by simply using $f = f_0 =$ maxwellian in the moment equations. If we also include f_1, then we obtain the so called "Navier Stokes" equations. In general, for f_1, $\vec{\nabla} \cdot \vec{\vec{P}} \neq \vec{\nabla} p$, i.e. the off diagonal elements of $\vec{\vec{P}}$ are $\neq 0$. Also, $\vec{q} \neq 0$. Using f_1 in our previous consistency conditions we can* obtain the following set of transport equations:

$$\partial n/\partial t + \vec{\nabla} \cdot n\vec{u} = 0 \qquad (1a)$$

$$nm \frac{d\vec{u}}{dt} = - \vec{\nabla} p + n\vec{F} + nm\mu \left[\frac{\vec{\nabla}(\vec{\nabla} \cdot \vec{u})}{3} + \nabla^2 \vec{u}\right] \qquad (2a)$$

$$\frac{3}{2} n \frac{dT}{dt} + nT \vec{\nabla} \cdot \vec{u} = - \vec{\nabla} \cdot \vec{q} \qquad (3a)$$

$$\vec{q} \equiv - \kappa \vec{\nabla} T$$

$$\mu \equiv C_\mu \lambda^2 \nu$$
$$\kappa \equiv C_k \lambda^2 \nu$$

where C_μ, C_k are numerical coefficients of order unity.

These equations are not "ideal" because they include viscosity and heat conduction.

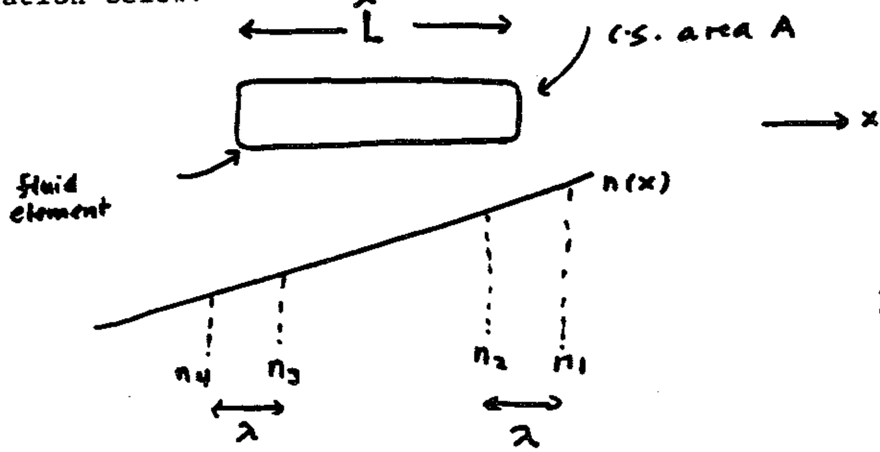
4. Physical meaning of pressure and viscosity

Meaning of pressure (particle point of view)

When particles with momentum mv_{th} embed themselves or escape from a fluid element, that element gains or loses momentum. We show that if there is a pressure differential across an element, then that difference manifests itself in a rate of change of average momentum of the

* assuming, of course, we have solved explicitly for f_1.

element. We also show that this force is proportional to $\vec{\nabla}p$. Consider the situation below:



$\lambda \ll \hat{L} \ll L$

Let $\vec{\nabla}n = \partial n/\partial x \neq 0$.

Then in a time ν^{-1}, particles at the edge a distance λ away enter or leave the element. Nett momentum gain is then

$$\Delta p)_+ = mv_{th} A\lambda (n_1 + n_2 - n_3 - n_4)$$

Since $\lambda \ll \hat{L}$, $n_1 \approx n_2$ and $n_3 \approx n_4$; also $n_2 = n_3 + |\partial n/\partial x|\hat{L}$ since $\hat{L} \ll$

$$\Rightarrow \quad \Delta p)_+ \approx \lambda A m v_{th} |\partial n/\partial x| \hat{L} .$$

Since this gain happens on time ν^{-1}, the <u>rate</u> of change of momentum $\sim \lambda A\hat{L} m v_{th} |\partial n/\partial x| \nu$.

This must be equated with mass x acceleration.

$$\therefore \lambda A\hat{L} m v_{th} (\partial n/\partial x)\nu = nmA\hat{L} \frac{du}{dt} .$$

Using $\lambda\nu \equiv v_{th}$ and $v_{th}^2 = T/m$, we have

$$nm\,\frac{du}{dt} \sim \left|\frac{\partial n}{\partial x}\right| T = \left|\frac{\partial p}{\partial x}\right|$$

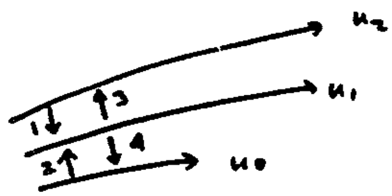
We have obtained order of magnitude agreement with $nm\partial\vec{u}/\partial t = -\vec{\nabla}p$.

Meaning of viscosity

Consider the sheared flow situation shown below.

For this flow, u_2 will drag on u_1, u_1 on u_0, and so on. What is the magnitude of the drag force?

Again, we compute the rate of change of directed momentum of an element in streamline u_1.



Random motions of particles such as 1 and 2 add momentum to u_1. Random motions 3 and 4 take away momentum.

$\therefore$ Nett increase in momentum in v^{-1}

$$= \quad nm\,(\overset{u_0}{\cancel{v_0}} + \overset{u_2}{\cancel{v_2}} - 2\overset{u_1}{\cancel{v_1}})$$

But $$u_2 = u_1 + \lambda \; |u_1'| + \frac{\lambda^2}{2} u_1''$$

$$u_o = u_1 - \lambda \; |u_1'| + \lambda^2/2 \; u_1''$$

$$\Rightarrow \quad \Delta p)_+ = nm \; \lambda^2 \; u_1''$$

Mass x accel. = $nm\lambda^2 \; \nu u_1''$

$$\Rightarrow \quad \frac{du_1}{dt} \sim \lambda^2 \; \nu u_1'' \; , \quad \Rightarrow \quad \tau \sim L^2/\lambda^2 \nu \; .$$

Note this agrees with the LHS and RHS (last term) of Eq. (2a).

Note: the above type of accounting may also be applied to understan' thermal conduction.

Final Note

Recall that the fluid equations were obtained by an expansion of the Boltzmann equation assuming

$$\nu\tau \gg 1, \; \lambda \ll L.$$

For the time scales obtained by us so far, namely

$$\tau_H \sim L/cs \; , \quad \tau_D \sim L^2/\lambda^2 \nu \; ,$$

we must make sure that (self-consistently)

$$\nu\tau_H \gg 1 , \quad \nu\tau_D \gg 1 .$$

This is true since

$$\nu\tau_H \sim L/\lambda \gg 1 , \quad \nu\tau_D \sim (L/\lambda)^2 \gg \nu\tau_H \gg 1 .$$