The Boltzmann equation for $f(\mathbf{x}, \mathbf{v}, t)$ is

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{F}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial f}{\partial t}^c.$$ 

This equation, in principle, tells us all we need to know about the gas, but, in practice, is difficult to solve. Since it contains more information than we need, we try to see if we can simplify the system by deriving equations for relevant quantities.

Relevant quantities could be:

- density, $n(\mathbf{x}, t) \equiv \int d^3v \ f$
- particle flux, $n_u(\mathbf{x}, t) \equiv \int d^3v \ \mathbf{v} f$
- pressure, $p(\mathbf{x}, t) \equiv \frac{1}{3} \int d^3v |\mathbf{v} - \mathbf{u}|^2 m f$
- temperature, from $p \equiv nT$.

So we begin by taking moments of the Boltzmann Eqn.
Multiply by \( \int d^3v \) and integrate each term.

Define \( \langle g \rangle \equiv \int d^3v \, g \)

then \( \langle f \rangle = n \),

\[
\langle \hat{v} \cdot \hat{v} f \rangle = \hat{v} \cdot \langle \hat{v} f \rangle = \hat{v} \cdot nu^+ ,
\]

\[
\langle \hat{v} \cdot \frac{\partial}{\partial \hat{v}} f \rangle = \langle \frac{\partial}{\partial v^+} \hat{v} f \rangle = \int_{s_v} ds^+ \cdot \hat{v} f = 0 ,
\]

where we have used Gauss' Theorem and assume that \( f \to 0 \) as \( |\hat{v}| \to \infty \),

and \( \langle \frac{\partial n}{\partial t} \rangle_c = 0 \), by particle conservation.

\[
\Rightarrow \frac{\partial n}{\partial t} + \hat{v} \cdot nu^+ = 0. \tag{1}
\]

Note that the evolution of \( n \), (zeroth moment of \( f \)), is governed by \( \hat{u} \) (first moment of \( f \)). So we take the first moment.

**First moment**

Multiply by \( \int d^3v \ \hat{v} \), integrate,

Define \( \hat{v} \equiv \hat{u}(\hat{x},t) + \delta \hat{v} \)

and note that \( \langle \delta \hat{v} \rangle \equiv 0 \) by definition; then,
\[ \langle \mathbf{v} f \rangle \equiv n u ; \]

\[ \langle \mathbf{v} \mathbf{v} \cdot \mathbf{v} f \rangle = \mathbf{v} \cdot \langle \mathbf{v} \mathbf{v} f \rangle \]

\[ = \mathbf{v} \cdot \{ \langle u \mathbf{u} f \rangle + \langle (u \delta \mathbf{v} + \delta \mathbf{v} u) f \rangle + \langle \delta \mathbf{v} \delta \mathbf{v} f \rangle \} \]

\[ = \mathbf{v} \cdot u u + \mathbf{v} \cdot \langle \delta \mathbf{v} \delta \mathbf{v} f \rangle ; \]

\[ \langle \mathbf{v} \Delta f \rangle = \langle \mathbf{v} \mathbf{v} \cdot \Delta f \rangle = \langle \nabla \cdot \mathbf{v} f \rangle - \langle f \mathbf{v} \cdot \nabla \mathbf{v} \rangle , \]

using the identity \[ \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{v} f = \nabla \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{v} f + f \nabla \frac{\partial}{\partial \mathbf{v}} \]

\[ = \int_{\text{surface integral}} - \mathbf{F} \langle f \rangle , \]

using Gauss' theorem and \[ \frac{\partial}{\partial \mathbf{v}} = \nabla \]

\[ = - n \mathbf{F} ; \]

\[ \langle \mathbf{v} \left( \frac{\partial f}{\partial t} \right) \rangle \equiv 0, \text{ by conservation of momentum}. \]

\[ m \frac{\partial}{\partial t} n u + n \mathbf{v} \cdot u u = - \nabla \cdot \mathbf{F} + n \mathbf{F} , \]  \hspace{1cm} (2)

where \[ \mathbf{F} \equiv m \langle \delta \mathbf{v} \delta \mathbf{v} f \rangle \] is the pressure tensor.

Note that the evolution of the 1st moment, \[ \mathbf{u} \], is governed by the 2nd moment \[ \langle \delta \mathbf{v} \delta \mathbf{v} f \rangle \].
We therefore need an evolution equation for the second moment and must take the second moment of the Boltzmann equation. However, we shall in general find that taking the nth moment will introduce the (n+1)th moment in our system of equations. Thus, we do not obtain closure. For example, the $m \frac{v^2}{2}$ moment (2nd moment) is

$$\frac{\partial}{\partial t} \left( nmu^2/2 + \frac{3}{2} p \right) + \nabla \cdot \left( nmu^2/2 + \frac{3}{2} p \right) \dot{u}$$

$$+ \nabla \cdot \ddot{p} \cdot \dot{u} = - \nabla \cdot \ddot{q} + n \ddot{F} \cdot \dot{u}$$

(3)

where $p \equiv \frac{1}{3} m \langle \delta v^2 f \rangle$, 

$$\ddot{q} \equiv \frac{1}{2} m \langle \delta v^2 \delta v_f \rangle,$$

Thus, the 2nd moment terms, proportional to $p$, are driven by 3rd moment terms, proportional to $\ddot{q}$.

In general, we need some closure scheme for the moment equations. One of these is the Chapman-Enskog scheme that assumes that collisions are dominant.

In terms of $p$, (1) - (3) can be recombined to restate (3) as

$$\frac{\partial}{\partial t} \left( \frac{3}{2} p \right) + \nabla \cdot \left( \frac{3}{2} p \dot{u} \right) + \nabla \cdot \ddot{p} \cdot \dot{u} = - \nabla \cdot \ddot{q}$$

(4)

(Note: $\ddot{F}$ drops out)