R1

Subsonic Fluid Response for $g \neq 0$: **Reduced eqns for $\partial_t \sim \sqrt{g} \to 0$**

$$\partial_t n + \vec{\nabla} \cdot (n\vec{u}) = 0 \qquad (1)$$

$$nM\, d\vec{u}/dt = -\vec{\nabla} p + nM\vec{g} \qquad (2)$$

$$dp/dt + \gamma p\, \vec{\nabla} \cdot \vec{u} = 0 \qquad (3)$$

Expect $\partial_t \sim \sqrt{g/L} \ll k c_s$

Expect convection $\frac{u}{L} \sim \partial_t$

Size of terms :

(1) $\partial_t n : \frac{nu}{L} \Rightarrow 1:1$

(3) $\partial_t : u/L \Rightarrow 1:1$

(2) $\cancel{nM} u\, \partial_t : \frac{p}{LnM} : \cancel{nM} g$

$\sim \frac{u\, \partial_t}{L} : \frac{c_s^2}{L^2} : \frac{g}{L}$

$\sim \partial_t^2 : (c_s/L)^2 : (g/L)$

$\epsilon^2 : 1 : \epsilon^2$

Lowest order

$$\vec{\nabla} p = 0 \implies p = p(t) \to P_0$$

$$\Rightarrow \boxed{\begin{aligned} &\partial_t n + \vec{\nabla}\cdot(n\vec{u}) = 0 \quad (1') \\ &\vec{\nabla}\cdot\vec{u} = 0 \quad (3') \end{aligned}}$$

$\leftarrow$ incompressible

$$nM\frac{d\vec{u}}{dt} = -\vec{\nabla} p_1 + nM\vec{g}$$

annihilate

$$\Rightarrow \vec{\nabla}\times(nM\,d\vec{u}/dt) = \vec{\nabla} n \times M\vec{g}$$

$$\Rightarrow \boxed{\vec{\nabla}\times(n\,d\vec{u}/dt) = \vec{\nabla} n \times \vec{g}} \quad (2')$$

$(1')$, $(2')$, $(3')$ New set for $\{n, \vec{u}\}$ 4 vars

<u>Reduced eqns</u> for low frequency.

p is eliminated.

There are sound waves here.

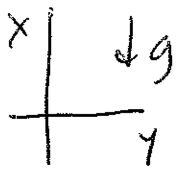
Linearized Red^d Eqns and RT mode

Equil $\quad \vec{\nabla} n \times \vec{g} = 0 \quad \Rightarrow n(x)$

$n(x)$ arbitrary

Perturb

$$\boxed{\begin{aligned} \partial_t \tilde{n} + \vec{\nabla}\cdot(n\tilde{\vec{u}}) &= 0 \\ \vec{\nabla}\cdot\tilde{\vec{u}} &= 0 \\ \vec{\nabla}\times(n\,\partial_t\tilde{\vec{u}}) &= \vec{\nabla}\tilde{n}\times\vec{g} \end{aligned}}$$



$\tilde{n} \to \tilde{n}(x)\, e^{iky - i\omega t}$; $\quad k_z = 0$ is ok

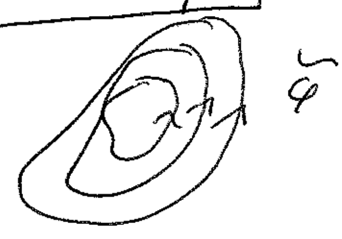
Note $\quad \hat{y}\cdot\vec{\nabla}\times n\tilde{\vec{u}} = -\partial_x(n\tilde{u}_z) = 0$

$\Rightarrow \boxed{\tilde{u}_z = 0}$

$\therefore$ only $\{\tilde{n}, \tilde{u}_x, \tilde{u}_y\}$

Now $\quad \vec{\nabla}\cdot\tilde{\vec{u}} = 0 \quad \Rightarrow \boxed{\tilde{\vec{u}} = \hat{z}\times\vec{\nabla}\tilde{\varphi}}$

$\tilde{\varphi}$ = stream function



$\Rightarrow \vec{\nabla}\times(n\hat{z}\times\vec{\nabla}\tilde{\varphi}) = \hat{z}\,\vec{\nabla}\cdot(n\vec{\nabla}\tilde{\varphi})$

$$\Rightarrow \boxed{\begin{aligned} &\partial_t \tilde{n} + \hat{z}\times\vec{\nabla}\tilde{\varphi}\cdot\vec{\nabla} n = 0 \\ &\partial_t \vec{\nabla}\cdot(n\vec{\nabla}\tilde{\varphi}) = g\partial_y \tilde{n} \end{aligned}}$$

$$\Rightarrow -i\omega\tilde{n} - ikn'\tilde{\varphi} = 0$$

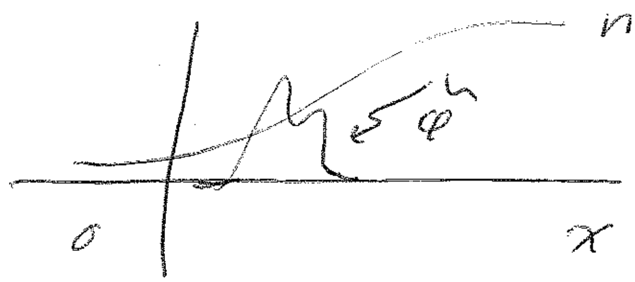
$$-i\omega\vec{\nabla}\cdot(n\vec{\nabla}\tilde{\varphi}) = +ikg\tilde{n}$$

Eliminate $\tilde{n}$ $\Rightarrow$

$$\boxed{\vec{\nabla}\cdot[n\vec{\nabla}\tilde{\varphi}] = \frac{gk^2}{\omega^2}n'\tilde{\varphi}}$$ eigenvalue eqn

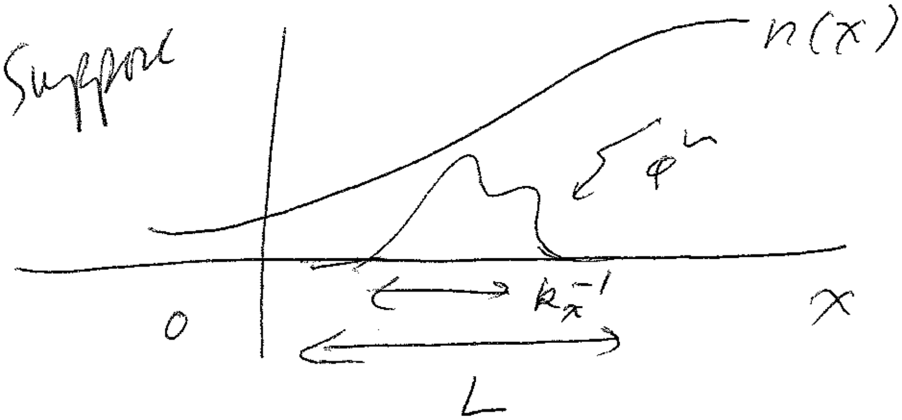
$$\Rightarrow \boxed{\tilde{\varphi}'' - k^2\tilde{\varphi} = \left(\frac{gk^2}{\omega^2}\right)\frac{n'}{n}\tilde{\varphi}}$$

$$\boxed{\tilde{\varphi}(|x|\to\infty) \longrightarrow 0}$$



Diffuse Profile Analysis (RT)

Suppose



let $n = n(x)$

Note: $k_y \to \infty \Rightarrow$ 0000 $\Rightarrow 1 \simeq -\frac{g\cancel{k}^2}{\omega^2}\frac{n'}{n}$

$\Rightarrow \omega^2 \simeq -\frac{g\, n'}{n}(x)$ can't be

but expect max instability where

$\frac{n'}{n}(x) \longrightarrow \max \Rightarrow \left(\frac{n'}{n}\right) \simeq \left(\frac{n'}{n}\right)_{max} - \left(\frac{n'}{n}\right)''\frac{(x-x_m)^2}{2}$

$$\therefore \boxed{\frac{\tilde\varphi''}{k^2} - \tilde\varphi \simeq \frac{g}{\omega^2}\left(\frac{n'}{n}\right)_m\left[1-\frac{s^2}{L_n^2}\right]\tilde\varphi} \quad s \equiv x - x_m$$

self cons $\Leftrightarrow k_x L_n \gg 1$

$$\Rightarrow \boxed{\tilde\varphi_{ss} = k^2\left\{1 + \left(\frac{g n'}{\omega^2 n}\right)_m\left[1-\frac{s^2}{L_n^2}\right]\right\}\tilde\varphi}$$

$\tilde\varphi(|x| \to \infty) \to 0$

- Parabolic Cylinder Eqn
- eigenvalue pblm

$\Rightarrow$ H.O. in QM, Hermite Fns

Try Lowest: $\hat{\varphi} \to \hat{\varphi}\, e^{-s^2/2\Delta^2}$

$$s \equiv (x - x_m)$$

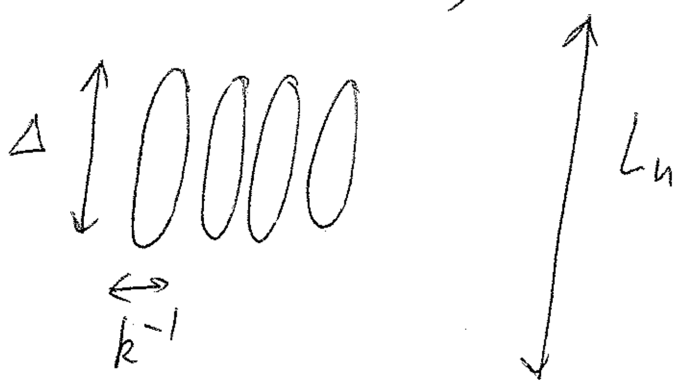
$$\hat{\varphi}_s = -\frac{s\hat{\varphi}}{\Delta^2}, \quad \hat{\varphi}_{ss} = -\frac{\hat{\varphi}}{\Delta^2} + \frac{s^2\hat{\varphi}}{\Delta^4}$$

$$\Rightarrow \boxed{\omega^2 \simeq \frac{-g(n'/n)_{max}}{(1+1/k^2\Delta^2)}} \quad \text{and} \quad \frac{k^2 L_n^2}{k^4\Delta^4} \simeq 1 + \frac{1}{k^2\Delta^2}$$

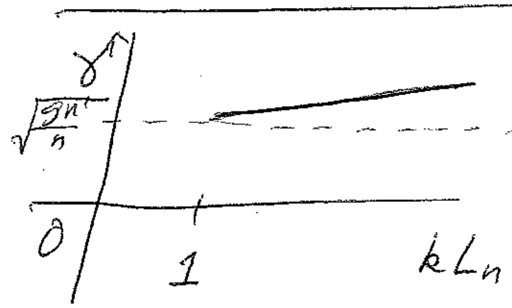
But demand $\Delta \ll L_n$

$$\Rightarrow k\Delta \simeq \sqrt{kL_n} \Rightarrow \Delta \simeq \sqrt{L_n/k}$$

with $kL_n \gg 1$, $k\Delta \gg 1$

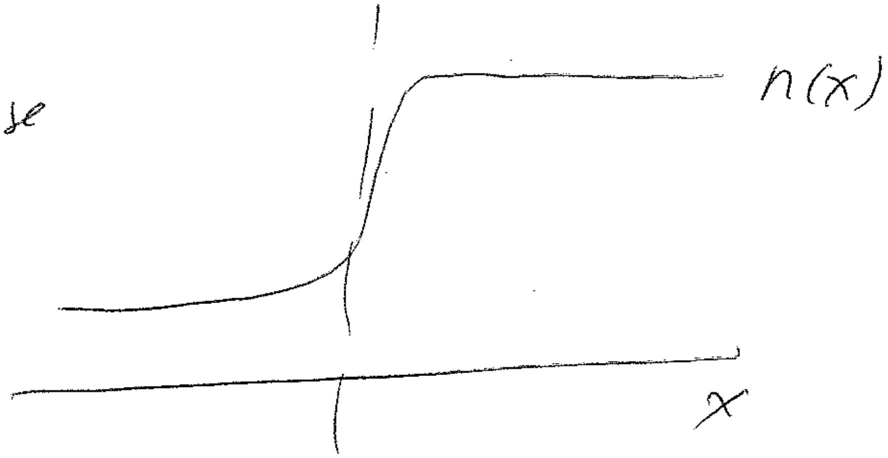


UNSTABLE for $n' > 0$



Sharp boundary analysis (RT)

Suppose



i.e. suppose $L_n \to 0$ (sharp change)

$\Rightarrow$ let $n(x) \to$ step function @ $x=0$

$\Leftrightarrow$ $k L_n \ll 1$ and $L_n \ll \Delta$

$$\boxed{\tilde{\varphi}'' - k^2 \tilde{\varphi} = \frac{g k^2}{\omega^2} \frac{n'}{n} \tilde{\varphi}}$$

$x \neq 0 \Rightarrow n' = 0 \Rightarrow \tilde{\varphi}'' - k^2 \tilde{\varphi} = 0$

$$\Rightarrow \tilde{\varphi} = \begin{cases} \hat{\varphi}_+ e^{-kx}, & 0 < x \\ \hat{\varphi}_- e^{+kx}, & x < 0 \end{cases}$$

Now $\displaystyle \frac{d}{dx}\left[n \frac{d\tilde{\varphi}}{dx} \right] \simeq \frac{g k^2}{\omega^2} \frac{dn}{dx} \tilde{\varphi}$

$$\int_{-\epsilon}^{\epsilon} dx \Rightarrow \left[n \frac{d\tilde{\varphi}}{dx} \right]_{-\epsilon}^{\epsilon} \simeq \frac{g k^2}{\omega^2} \left[\left[n \tilde{\varphi} \right]_{-\epsilon}^{\epsilon} - \cancelto{0}{\int_{-\epsilon}^{\epsilon} dx\, n \tilde{\varphi}'} \right]$$

$$\Rightarrow \left[n\tilde{\varphi}'\right]_-^+ = \frac{gk^2}{\omega^2}\left[n\tilde{\varphi}\right]_-^+$$

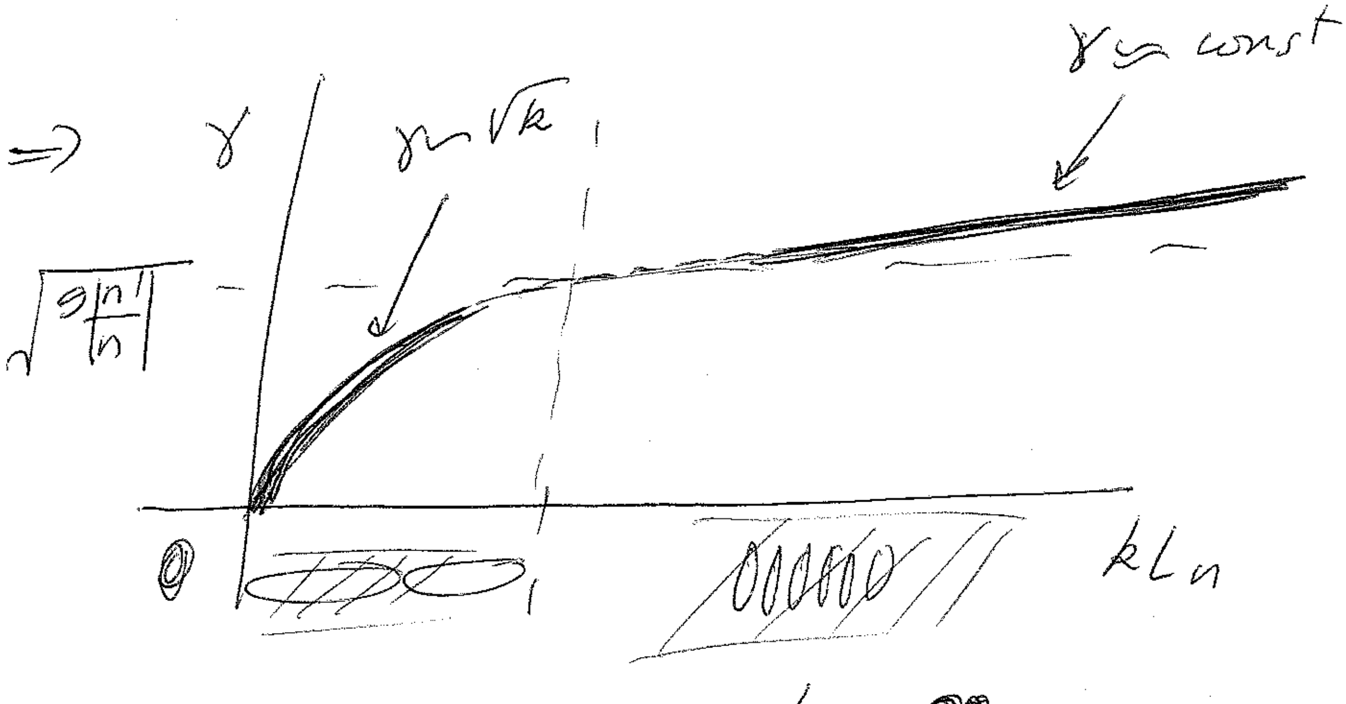
also can show $\left[\tilde{\varphi}\right]_-^+ = 0$, by a 2nd $\int$

$$\Rightarrow -n_+k - n_-k = \frac{gk^2}{\omega^2}\left[n_+ - n_-\right]$$

$$\Rightarrow \boxed{\omega^2 = -gk\left(\frac{n_+ - n_-}{n_+ + n_-}\right)}$$

Need $k\cancel{\Delta}_n \ll 1$ and $\Delta \cancel{\gg} L_n$, $\Delta k \sim 1$

$$\Rightarrow kL_n \ll 1$$

$\Rightarrow$ 

γ peaks weakly as $k \to \infty$