Reduced eqns in plasma w $B_z \to \infty$

(1) $$\partial_t n + \vec{\nabla}\cdot(n\vec{u}) = 0$$

(2) $$n M d_t \vec{u} = -\vec{\nabla}(p + B^2/8\pi) + \vec{B}\cdot\vec{\nabla}\vec{B}/4\pi$$

(3) $$d_t p + \gamma p \vec{\nabla}\cdot\vec{u} = 0$$

(4) $$\partial_t \vec{B} = \vec{\nabla}\times(\vec{u}\times\vec{B})$$

Also $\vec{\nabla}\cdot\vec{B} = 0$ (5)

Consider system $\ni \underbrace{\dfrac{B_z^2}{8\pi} \gg p \sim nMu^2}_{\text{low } \beta}$

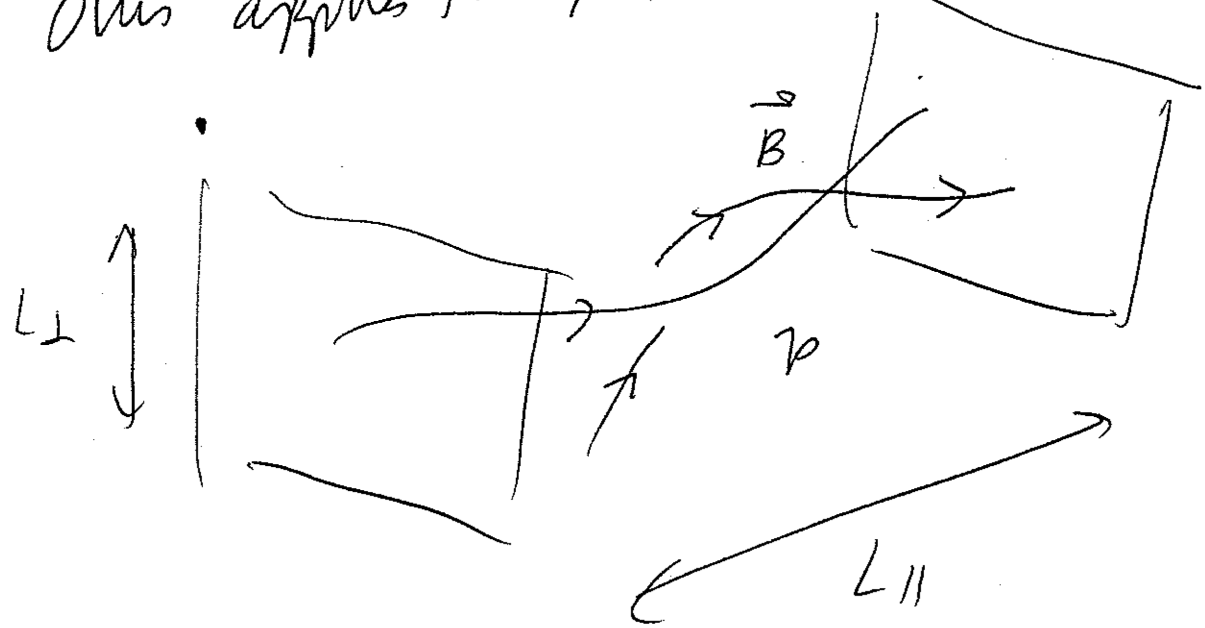
MHD $\Rightarrow \exists \left(\dfrac{V_A}{L}\right)^2$ and $\left(\dfrac{C_s}{L}\right)^2$ waves.

But also, $\dfrac{V_A}{L}$ → $V_A/L_\perp$ ← M'sonic

→ $V_A/L_\parallel$ ← Alfven

We want sound waves and Alfven waves in our description but want to be low frequency w.r.t. magsonic wave, ie, $\dfrac{V_A}{L_\perp} \gg \dfrac{C_s}{L_\parallel} \sim \dfrac{V_A}{L_\parallel}$

This applies to systems like



$L_\parallel \gg L_\perp$, $|\vec{B}_\perp| \ll B_z$

∴ our ordering is

$$\frac{B_z^2}{8\pi} \gg \frac{|\vec{B}_\perp|^2}{8\pi} \sim p \sim nmu^2$$

and $L_\perp \gg L_\parallel$. Also $\partial_t \sim \frac{u}{L_\perp}$

Thus, in ①–④, to lowest order, B_z and $|\vec{v}_\perp|$ must be kept but all else is small.

∴ lowest order ⟹ $\boxed{\vec{\nabla}_\perp\left(\frac{B_z^2}{8\pi}\right) = 0}$ (2′)

$\Rightarrow B_z = B_z(z,t)$

other lowest order eqns are

$$\partial_t n + \vec{\nabla}_\perp \cdot (n\vec{u}) = 0 \qquad (1')$$

$$\partial_t p + \vec{u}\cdot\vec{\nabla}_\perp p + \gamma \vec{\nabla}_\perp \cdot \vec{u} = 0 \qquad (3')$$

$$\partial_t (\hat{z} B_z) = \vec{\nabla}_\perp \times (\vec{u} \times \hat{z} B_z) \qquad (4')$$

$$\vec{\nabla}_\perp \cdot \vec{B}_\perp + \partial_z B_z = 0 \qquad (5')$$

Now (4') $\Rightarrow \hat{z}(\partial_t B_z) = \hat{z} B_z \vec{\nabla}_\perp \cdot \vec{u}$

$$\Rightarrow \frac{\partial_t B_z(z,t)}{B_z} = \vec{\nabla}_\perp \cdot \vec{u}_\perp$$

There are several ways to show that $\vec{\nabla}_\perp \cdot \vec{u}_\perp = 0$. Note this $\Rightarrow \partial_t B_z = 0 \Rightarrow \nabla_\perp \cdot u_\perp = 0$. Note that LHS does not depend on $\vec{X}_\perp$, RHS does. Thus, one way, suppose system is bounded in $|\vec{X}_\perp|$; then, $|\vec{X}_\perp| \to \infty \Rightarrow$ RHS $\to 0 \Rightarrow$ LHS $= 0 \Rightarrow$ RHS $= 0$.

$$\therefore \boxed{\vec{\nabla} \cdot \vec{u}_\perp = 0} \quad (6) \qquad \text{and } B_z = B_z(z)$$

Now (6) $\Rightarrow$ $\boxed{\vec{u}_\perp = \vec{z} \times \vec{\nabla} \varphi}$ (6')

stream function

use these results in (1'), (3') $\Rightarrow$

$$\partial_t n + \vec{z} \times \vec{\nabla}_\perp \varphi \cdot \vec{\nabla}_\perp n = 0 \quad (1'')$$

$$\partial_t p + \vec{z} \times \vec{\nabla}_\perp \varphi \cdot \vec{\nabla}_\perp p = 0 \quad (3'')$$

(assumes $\frac{u_\perp}{L_\perp} \gg \frac{u_\parallel}{L_\parallel}$)

We now go to first order in (2).

To be precise, let $\frac{|\vec{B}_\perp|}{B_z} \sim \epsilon$, $\frac{|\nabla_\parallel|}{|\nabla_\perp|} \sim \epsilon$.

Then, LHS (2) $\sim O(\epsilon^2)$. $\Rightarrow$

$$\vec{\nabla}_\perp B_{z1} = 0 \quad (2'')$$

We can set $B_{z1} = 0$ w/out loss.

We simplify here as follows: we assume that B_z = const to lowest order, ie, $B_z = B_0$, not dependent on z. This is OK, can be fixed later if needed. (It cuts out weak mirror geometry.) BT

In this case, (5') to lowest non-vanishing order becomes

$$\vec{\nabla}_\perp \cdot \vec{B}_\perp = 0 \quad (5'')$$

$$\Rightarrow \boxed{\vec{B}_\perp = \hat{z} \times \vec{\nabla}_\perp \psi}$$ introduce "magnetic streamfunction"

Now we go to (2) to $O(\epsilon^2) \Rightarrow$

$$nM\, d_t \vec{u} = -\vec{\nabla}_\perp (P_2 + B_z B_{z2}) + \vec{J}_1 \times \vec{B}_\perp$$

where $\vec{J}_1 = (\vec{\nabla} \times \vec{B})_1$

Before solving for P_2, B_{z2}, annihilate by $\hat{z} \cdot \vec{\nabla}_\perp \times \Rightarrow$

$$\hat{z} \cdot \vec{\nabla}_\perp \times (nM\, d_t \vec{u}) = \vec{B} \cdot \vec{\nabla} J_{z1}$$

where $\vec{B} \cdot \vec{\nabla} = B_z \partial_z + \vec{B}_\perp \cdot \vec{\nabla}_\perp$

and $J_{z1} = \hat{z} \cdot (\vec{\nabla} \times \vec{B})_1 = -\nabla_\perp^2 \psi$

$$\Rightarrow \quad \hat{z}\cdot\vec{\nabla}_\perp\times\left(nM\,d_t\vec{u}\right) = \vec{B}\cdot\vec{\nabla}\,\nabla_\perp^2\psi \qquad (2''')$$

Finally, consider (4) to 2nd order

$$\partial_t\vec{B}_\perp = \vec{\nabla}_\perp\times\left(\overbrace{\vec{u}\times\vec{B}_\perp}^{(\hat{z}\times\vec{\nabla}\varphi)\times\vec{B}_\perp}\right) + \vec{\nabla}\times\left(\overbrace{\vec{u}\times\hat{z}B_z}^{(\hat{z}\times\vec{\nabla}\varphi)\times\hat{z}}\right)$$

$$\Rightarrow \quad \hat{z}\times\vec{\nabla}_\perp\partial_t\psi = \hat{z}\times\vec{\nabla}_\perp(\vec{B}_\perp\cdot\vec{\nabla}\varphi) + \hat{z}\times\vec{\nabla}_\perp(B_z\partial_z\varphi)$$

$$\Rightarrow \quad \boxed{\partial_t\psi = \vec{B}\cdot\vec{\nabla}\varphi,} \qquad \vec{B}\cdot\vec{\nabla} \equiv B_z\partial_z + \vec{B}_\perp\cdot\vec{\nabla}_\perp$$

$$(4'')$$

We are now done. We have a closed system of equations for $\{n, \varphi, \psi\}$ from (1''), (2'''), (4'') summarized as follows

$\{n, \varphi, \psi\}$ system

$$\partial_t n + \vec{u}_\perp \cdot \vec{\nabla}_\perp n = 0 \qquad (A)$$

$$\hat{z} \cdot \vec{\nabla}_\perp \times (n M d_t \vec{u}_\perp) = \vec{B} \cdot \vec{\nabla} \nabla_\perp^2 \psi \qquad (B)$$

$$\partial_t \psi = \vec{B} \cdot \vec{\nabla} \varphi \qquad (C)$$

Definitions

$$\vec{u}_\perp \equiv \hat{z} \times \vec{\nabla}_\perp \varphi, \quad \vec{B}_\perp \equiv \hat{z} \times \vec{\nabla}_\perp \psi,$$

$$\vec{B} \cdot \vec{\nabla} \equiv B_z \partial_z + \vec{B}_\perp \cdot \vec{\nabla}_\perp$$

$$B_z = B_0 = \text{const}$$

Can add "g"

$$\hat{z} \cdot \vec{\nabla}_\perp \times (n M \partial_t \vec{u}) = \hat{z} \cdot \vec{\nabla}_\perp n \times \vec{g} + \vec{B} \cdot \vec{\nabla} \nabla_\perp^2 \psi$$