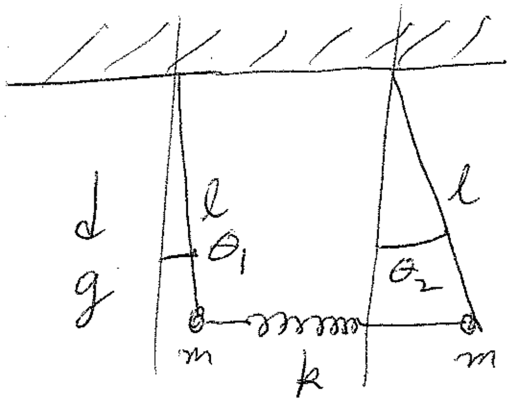
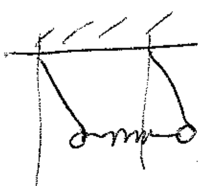
Coupled oscillators – Approximate solution for disparate Normal Mode frequencies

$$\boxed{\begin{aligned} m\ell\ddot{\theta}_1 &= -mg\theta_1 + k\ell(\theta_2-\theta_1) \\ m\ell\ddot{\theta}_2 &= -mg\theta_2 - k\ell(\theta_2-\theta_1) \end{aligned}}$$

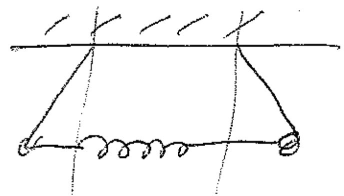


There are 2 normal modes

$e^{-i\omega t} \Rightarrow$



OR

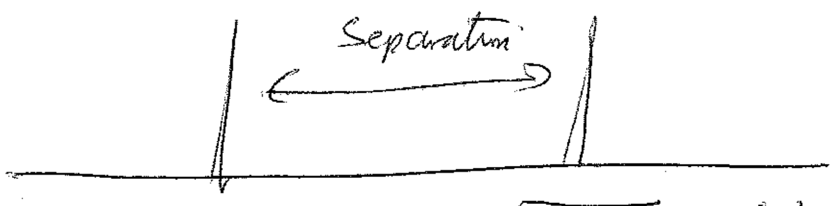


$\omega_1^2 = g/\ell$ $\qquad\qquad$ $\omega_2^2 = g/\ell + 2k/m$

Suppose $g/\ell \ll k/m \Rightarrow \omega_1^2 = g/\ell$

$\omega_2^2 \simeq 2k/m \gg \omega_1^2$

$\Rightarrow$ Spectrum

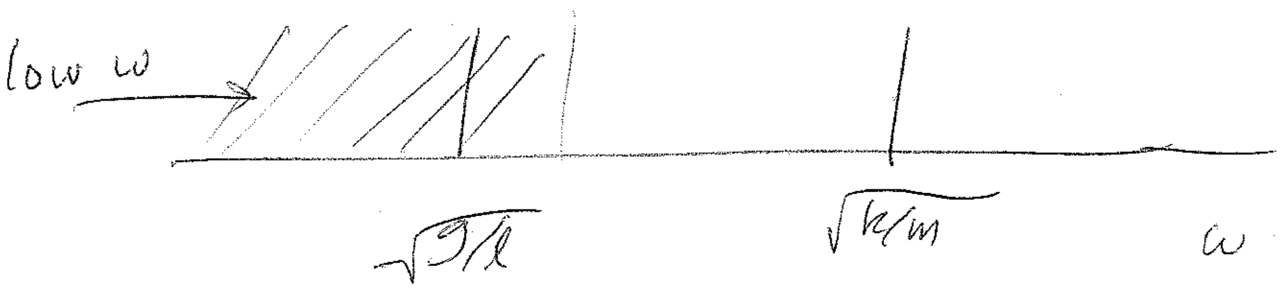


Approximate solution ("reduced eqns")

$$\ddot{\theta}_1 = -\frac{g}{\ell}\theta_1 + \frac{k}{m}(\theta_2-\theta_1) \qquad (1)$$

$$\ddot{\theta}_2 = -\frac{g}{\ell}\theta_2 - \frac{k}{m}(\theta_2-\theta_1) \qquad (2)$$

Suppose we are interested only in "low frequencies", ie, assume we want $\partial/\partial t \ll \sqrt{k/m}$



Then, "scaling" ①, we have

$$\ddot{\theta}_1 = -\frac{g}{l}\theta_1 + \frac{k}{m}(\theta_2 - \theta_1)$$

$$\longrightarrow \quad \omega^2\theta_1 : \frac{g}{l}\theta_1 : \frac{k}{m}\theta_2 : \frac{k}{m}\theta_1$$

These two are <u>smaller</u> than this

we don't know $\theta_2 : \theta_1$, so leave θ_2 in tact. Thus, if $\omega^2 \ll (k/m)$, $g/l \ll k/m$

$$\Rightarrow \quad 0 \simeq \frac{k}{m}\left(\theta_2^{(0)} - \theta_1^{(0)}\right) \qquad (1')$$

likewise from ②,

$$0 \simeq -\frac{k}{m}\left(\theta_2^{(0)} - \theta_1^{(0)}\right) \qquad (2')$$

where $\theta_i = \theta_i^{(0)} + \theta_i^{(1)} + \theta_i^{(2)} + \cdots$ etc

$$\Rightarrow \boxed{\theta_2^{(0)}(t) \simeq \theta_1^{(0)}(t) \,, \text{ lowest order.}}$$

∴ eqns to first order are

$$\ddot{\theta}_1^{(0)} = -\frac{g}{\ell}\theta_1^{(0)} + \frac{k}{m}\left[\theta_2^{(1)} - \theta_1^{(1)}\right] \quad (3)$$

$$\ddot{\theta}_2^{(0)} = -\frac{g}{\ell}\theta_2^{(0)} - \frac{k}{m}\left[\theta_2^{(1)} - \theta_1^{(1)}\right] \quad (4)$$

These are eqns for $\theta_1^{(1)}$ & $\theta_2^{(1)}$.

However, they are <u>inhomogeneous</u> eqns for $\theta_1^{(1)}$ & $\theta_2^{(1)}$! Must find annihilator(s) as solns may not always exist. One annihilator [that kills the "largest term"] is to

<u>add</u> (3) + (4) $\Rightarrow$ $\ddot{\theta}_1^{(0)} + \ddot{\theta}_2^{(0)} = -\frac{g}{\ell}\left(\theta_1^{(0)} + \theta_2^{(0)}\right)$

$\theta_1^{(0)} = \theta_2^{(0)} \Rightarrow \ddot{\theta}_1^{(0)} = -\frac{g}{\ell}\theta_1^{(0)}$

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$$\Rightarrow \boxed{\begin{array}{l}\theta_1^{(0)}(t) = \text{Const}\; e^{-i\omega_0 t} \\ \text{where } \omega_0 = \pm\sqrt{g/\ell}\end{array}}$$

and, of course, $\theta_2^{(0)}(t) = \theta_1^{(0)}(t)$.

We have obtained MORE info on the lowest order soln by annihilation

Note we have obtained the low frequency normal mode.

First order

Now the first order equation is

$$\frac{k}{m}\left[\theta_2^{(1)} - \theta_1^{(1)}\right] = \ddot{\theta}_1^{(0)} + \frac{g}{\ell}\theta_1^{(0)}$$

But RHS $= 0 \Rightarrow \theta_2^{(1)} = \theta_1^{(1)}$

etc.

Self consistent check

we assumed $|\ddot{\theta}_1| \ll \frac{k}{m}|\theta_1|$

we found $\theta_1 \simeq \text{const}\ e^{-i\omega_0 t}$, $\omega_0^2 \simeq \frac{g}{\ell}$

$\therefore$ self consistent $\Leftrightarrow$

$$\omega_0^2 \ll (k/m) \Rightarrow \boxed{\frac{g}{\ell} \ll \frac{k}{m}}$$

which is the assumption