$$\omega_o^2 + \left(\frac{\partial v^{(o)}}{\partial x}\right)^2 = [\pi/Y(x)]^2 ,$$

as well as the η dependence of w (sinusoidal). All that the device has amounted to in this case, of course, is factoring out (from u) a fast varying function of x, but the use of the exponential representation has led to that procedure in a natural and systematic way.

We complete our list with the simple Principle of Mathematical Nonsense: if, in the course of an asymptotological analysis, a mathematically nonsensical expression appears, this indicates that the asymptotology has not been done correctly or at least not carried out fully (although even incomplete it may be satisfactory for one's purposes). One may come upon expressions such as 0/0, divergent sums or integrals, singular functions, etc., and whether they are to be considered nonsensical sometimes depends on the use they are to be put to. In the just discussed membrane vibration problem the first instance of mathematical nonsense was the disappearance in the limit of the region over which the partial differential equation was to be solved, the second was perhaps the dependence of ω on x, and the third was the response to this, the use of a singular (delta) function.

Frequent in asymptotological analyses is the occurrence of phenomena on different scales of distance or time. The HCE problem is a well known case (as Grad has just pointed out), since if f is not prescribed Maxwellian at the initial instant, there is a relatively short period of time (the order of a collision time) during which f becomes

Maxwellian, while the five moments remain approximately constant, and a relatively long period (of order λ times as long) during which the five moments (hydrodynamic variables) vary but f maintains its Maxwellian form. For an extremely simple example of the same type, consider the familiar electric circuit equation $V = RI + L\dot{I}$, where the voltage $V(t)$ is an imposed function of time, the current $I(t)$ is to be found, the resistance R and the inductance L are positive constants, and we choose to examine the limit $L \rightarrow 0$. Treating $L\dot{I}$ as if it were known, we immediately obtain a recursion formula for I,

$$\begin{aligned} I &= R^{-1}V - R^{-1}L\dot{I} \\ &= \frac{1}{R}\left[V - \frac{L}{R}\dot{V} + \left(\frac{L}{R}\right)^2 \ddot{V} - \left(\frac{L}{R}\right)^3 \dddot{V} + \ldots\right], \end{aligned} \qquad (13)$$

which is fine except for not in general satisfying the arbitrary initial condition on I natural for the original first order differential equation. For short times (of order L) $\dot{I}$ is large and V approximately constant, so that the difference of I from its quasi-equilibrium value V/R decays like $\exp(-Rt/L)$; after this transient has died out (13) holds. Incidentally, the expression in brackets in (13) is just like the Taylor expansion in powers of L of V evaluated at the argument $t - L/R$ except for a factor of $(n - 1)!$ in the denominator of the n-th term, which shows that the asymptotic series (13) for I cannot be expected to converge even if V is analytic (which does not stop it from being very useful).

In phenomena with behavior on two different time scales there is a widely pertinent distinction to be observed between finite conservative systems on the one hand and infinite or dissipative systems on the other. For instance the well known problem of the harmonic oscillator with slowly varying coefficient of restitution,[12] $\ddot{x} + k(\epsilon t) x = 0$, is an example of the first kind; on the short (finite) time scale k is approximately constant and the oscillator simply oscillates steadily, while on the long $(\sim \epsilon^{-1})$ time scale the frequency and amplitude of the oscillation vary in response to the variation in k. Contrast with this the behavior of the dissipative electric circuit, where only initially the current I varies on the short time scale, swooping toward its quasi-steady value. The HCE example shows that a conservative system can act the same way so long as it is infinite; in this case the decay comes about by a process of "phase mixing," and is possible because the Poincaré recurrence time is infinite.

The asymptotic separation of time scales is the basis for an exciting recent approach in statistical mechanics.[13] Typically one obtains equations for the one-particle and the two-particle distribution functions f_1 and f_2 for a gas of appropriate characteristics, and finds f_1 can vary only slowly, but f_2 can vary quickly so as to phase-mix towards a quasi-steady distribution as t gets large on the short time scale while remaining small on the long time scale. The limiting distribution f_2 is a functional of f_1, which when substituted into the equation for $\dot{f}_1$ leads to a closed "kinetic equation" for f_1. The irreversibility (timewise) of this kinetic equation comes about in a natural way, in that the limiting f_2 depends on which direction t is taken to the limit (on the short time scale), whether to plus or to minus infinity. It is a major triumph of this approach that the "Stosszahlansatz" can for the first time be actually derived (under moderate smoothness assumptions).

To return to the conservative case, I am glad to take the opportunity of advertising a recent paper[14] in which I have elaborately worked out the asymptotic theory of finite systems of ordinary differential equations depending on a small parameter ϵ which to lowest order have all solutions periodic. Applied to Hamiltonian systems the theory leads to the existence of adiabatic invariants which are constant (integrals) to all orders in ϵ.

We are all familiar with those rather unsatisfactory research papers in which the author makes a series of largely arbitrary *ad hoc* approximations throughout, often dubious without (sometimes even with) the author's intuitive grasp of the situation. These "ad-hoaxes" have their place and utility, but how much more desirable and convincing is a properly worked out and elegant asymptotological treatment, with any arbitrary assumptions (like remarkable coincidences in a well constructed mystery story) made openly and aboveboard right at the beginning where anyone can assess their merits for himself, and with the later development unfolding naturally and inexorably once a definite problem and the limit in which it is to be considered have been settled upon!

The art of asymptotology lies partly in choosing fruitful limiting cases to examine--fruitful first in that the system is significantly simplified and second in that the results are qualitatively enlightening or quantitatively descriptive. It is also an art to construct an appropriate generic description for the asymptotic behavior of the solution desired. The scientific element in asymptotology resides in the nonarbitrariness of the asymptotic behavior and of its description, once the limiting case has been decided upon.

Molière has one of his characters observe that for more than forty years he has been talking prose without knowing it. It is doubtful that he benefited from the discovery, but I hope that you will be more fortunate and not disappointed in having by now discovered that asymptotology is what you have been practicing all along!

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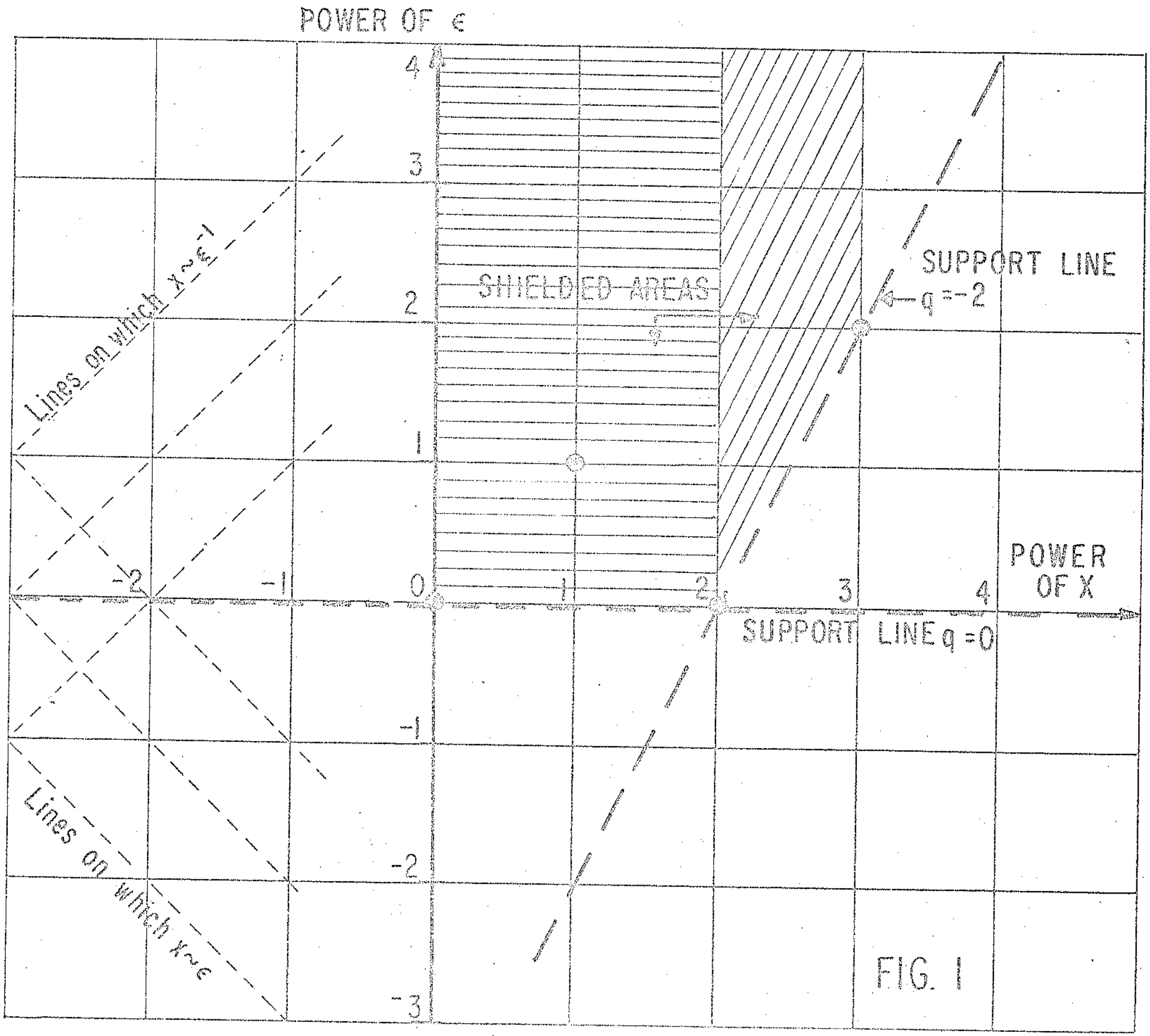

POWER OF ϵ
POWER OF X
SHIELDED AREAS
SUPPORT LINE
q = -2
SUPPORT LINE q = 0
Lines on which $x \sim \epsilon^{-1}$
Lines on which $x \sim \epsilon$

FIG. 1