information,* would "complicate" it if we were not able to recover sufficient additional conditions to make up for the information lost.

The basic way systems simplify is by the neglect of terms, as stated earlier. But it commonly happens that the relative asymptotic magnitude of two terms to be compared depends upon some knowledge not yet available or on some assumption or decision not yet made. According to the (sixth) Principle of Maximal Balance (or of Maximal Complication**), for maximal flexibility and generality we should keep both terms, i.e. we should allow for the possibility or assume that they are comparable. In the case of incomplete knowledge this is mere prudence; any term in an equation definitely smaller in order of magnitude than another term may be considered negligible, but no term should be neglected without a good reason. In the case of a pending assumption or decision, the desire to balance two such competing terms helps to determine the choice.

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*Use of this terminology is justified even from the technical viewpoint of information theory, suggesting the possibility of assigning a measure to the decrease in the number of solutions occurring in a limit.

**Partly as a consequence of Professor Friedrichs' comment at the conclusion of my lecture, I now feel that "Minimal Simplification" is more appropriate here.

The most widely applicable and hence most informative ordering is that which simplifies the least, maintaining a maximal set of comparable terms. Quite often there is more than one possible maximal set of terms, with no set including all terms of any other. (Sets of terms form a lattice ordered by inclusion.) Each maximal set corresponds to different asymptotic behavior. The solutions may split up according to which behavior they have (second way of simplifying), as with the cubic, or each solution may exhibit a variety of different behaviors, in different regions, as with a boundary layer phenomenon.

For instance in the case of the cubic equation, how could we know that two solutions are finite and one of order ϵ^{-2}? Put another way, why did we not assume the first and third terms to be the dominant ones, or the second and third, or so on? In this particular case there is an easy answer: if we had, we would have obtained a "solution" for which the neglected terms were not in fact negligible compared to the supposed dominant terms, i.e. the "solution" found would not have been self-consistent. But suppose there were several more terms, would we have had to try every pair? (Or suppose there were two independent small parameters δ and ϵ instead of only one.) Clearly, no matter which terms are dominant x will behave predominantly as some power of ϵ. We therefore assume the general representation $x \approx a\epsilon^q$ and wonder what value of q to take. One might in fact choose arbitrarily any value for q but will then generally find that for finite a only one term of (2) dominates, which is nonsensical, so that $a = \infty$ (if it was the constant

term), which is not legitimate, or else $a = 0$ (if it was one of the others), which, if more legitimate, is certainly no more useful. A value of q will only be "proper" if we end up with a representation which is "maximally complicated" in that it really consists of one term $a\epsilon^q$ instead of "no terms" such as 0 or ∞. If we put $x \approx a\epsilon^q$ into (2) the successive terms vary as ϵ to the respective powers $3q + 2$, $2q$, $q + 1$, 0, and it is easy to see that only $q = 0$ or $q = -2$ make two (or more) powers equal minima.

On the side it might be of interest to mention a graphical method of finding the proper values of q which apparently goes back to Newton. It is hardly needed in the present simple illustration but can be a great time-saver in more involved examples (also those of higher dimensionality). We plot each term of (2) as a point on a graph, the abscissa being the exponent of x and the ordinate that of ϵ (see four heavy points in figure 1); the coefficient is ignored so long as it is not zero. The specification of a definite relationship between x and ϵ (i.e. of a definite value of q) leads to the identification of the asymptotic behavior of all terms (present or not) corresponding to points which are on a common line with a definite slope. Thus for $x \sim \epsilon$ all points on the same down-slanting (from left to right) 45° line correspond to a common asymptotic behavior, while for $x \sim \epsilon^{-1}$ the same holds for up-slanting 45° lines (see dotted lines). Since the smaller the power of ϵ the larger the term, we seek lines passing through (at least) two graphed points and having no graphed points below them. We may think of finding the lower

convex support lines of the set of graphed points, perhaps kinesthetically by imagining pushing a line up from below until it first hits a graphed point and then rotating it around that point until it next hits a second graphed point. It is immediately apparent from figure 1 that there are just two such lines and that they correspond to $q = 0$ and $q = -2$ (see heavy dashed lines). It is also clear that the point (1, 1), like all points in a semi-infinite vertical strip (see horizontally shaded area), are "shielded" by the points (0, 0) and (2, 0) and can never be on a support line; it is indeed obvious that ϵx is negligible with respect to either x^2 or 4 no matter how x varies with ϵ. Similarly there is a semi-infinite vertical strip shielded by the points (2, 0) and (3, 2) (see diagonally shaded area). In more complicated cases we can thus exclude terms wholesale from competition.

To return to our proper business, illustration of the Principle of Maximal Complication, consider the problem of finding the lowest frequency of vibration and the corresponding form of vibration of a uniform membrane stretched between two close wires lying in a plane, one of which we take straight for simplicity. The equation for the standing vibration of a membrane is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \nu^2 u = 0 , \qquad (6)$$

where u is the displacement normal to the (x, y) plane, which is the rest plane of the membrane (the plane containing the wires), and ν is the

frequency of vibration of the mode. Let the equations of the wires in the (x, y) plane be $y = 0$ and $y = \epsilon Y(x)$, where ϵ of course is the small parameter of closeness. We may suppose $Y(x_1) = Y(x_2) = 0$ so as to have to consider only the finite region $x_1 < x < x_2$, $0 < y < \epsilon Y(x)$. Imposing the condition $u = 0$ on the boundary of this region and (6) inside the region, we have an eigenvalue problem for the lowest eigenvalue ν and its corresponding eigenfunction u. This is one common type of asymptotic problem, asymptotic rather than "perturbational" in that there is no limit problem because the region of interest disappears in the limit. The remedy for this is well known;[1] we rescale the variables appropriately, in this case introducing $\eta \equiv \epsilon^{-1} y$ so that the region in the (x, η) plane becomes $x_1 < x < x_2$, $0 < \eta < Y(x)$, and (6) becomes

$$\frac{\partial^2 u}{\partial x^2} + \epsilon^{-2} \frac{\partial^2 u}{\partial \eta^2} + \nu^2 u = 0 \quad . \tag{7}$$

Taking the asymptotic behavior of each term at its face value (but remembering that ν is not yet determined), we deem the first term negligible compared to the second, and (by the Principle) assume $\nu^2 \sim \epsilon^{-2}$ to balance the second and third terms. Introducing $\omega \equiv \epsilon \nu$ we write (7) as

$$\frac{\partial^2 u}{\partial \eta^2} + \omega^2 u = - \epsilon^2 \frac{\partial^2 u}{\partial x^2} \quad . \tag{8}$$

To lowest order we neglect the right side of (8), whereupon x degenerates from an independent variable to a mere parameter. The really proper treatment at this point, by the Principle of Recursion, would be to treat the right side of (8) as known, solve for u on the left in the form of an integral representation (involving the simple, well known, explicit Green's function), and try to obtain u iteratively. Instead we shall do something similar but simpler, more or less paralleling the lowest order version of the proper treatment. For each x we have, to lowest order, a simple eigenvalue problem with lowest eigenstate $u = A \sin(\pi\eta/Y)$ and eigenvalue $\omega = \pi/Y$. But ω so defined depends on x, which is impermissible, so we take $A(x)$ to be a Dirac delta function, the location of whose singularity we take to be at the maximum of $Y(x)$ in order to have the smallest ω; for simplicity we assume the maximum of Y to be unique and to occur at $x = 0$. In a sense we have now solved the problem originally posed, but since our answer is singular it is not entirely satisfactory (see the next and final Principle to be formulated). Indeed, since our "solution" is singular in its x dependence, we ought to worry whether our earlier neglect of $\epsilon^2 \frac{\partial^2 u}{\partial x^2}$ was justified, and we might well be curious anyway about the true detailed x dependence which we have cavalierly expressed as a delta function. Since the significant behavior occurs near $x = 0$ we introduce $\xi = \delta^{-1}x$, where δ is a small parameter to be determined (related to ϵ). We also write $\omega = \omega_o + \hat{\omega}$, where $\omega_o = \pi/Y(0)$ and $\hat{\omega}$ is small. Since

$\frac{\partial^2 u}{\partial \eta^2} \approx - \pi^2 Y(x)^{-2} u$, from (8) we obtain

$$\left[\frac{\pi^2}{Y(\delta \xi)^2} - \omega^2 \right] A \approx \frac{\epsilon^2}{\delta^2} \frac{d^2 A}{d\xi^2} \quad . \tag{9}$$

Let $Y(\delta \xi) = Y(0) + \frac{1}{2} Y''(0) \delta^2 \xi^2 + \ldots$ with $Y''(0) < 0$, whereupon this becomes

$$\left[- \frac{\pi^2 Y''(0)}{Y(0)^3} \delta^2 \xi^2 - 2 \omega_o \tilde{\omega} \right] A \approx \frac{\epsilon^2}{\delta^2} \frac{d^2 A}{d\xi^2} \quad . \tag{10}$$

According to the Principle of Maximal Complication we choose the as yet undetermined asymptotic behaviors so as to keep all the terms in the equation and are thus led to take $\delta = \epsilon^{1/2}$ and $\tilde{\omega} = \epsilon^{-1} \hat{\omega}$, obtaining

$$\frac{d^2 A}{d\xi^2} + \frac{\pi}{Y(0)} \left[\frac{\pi Y''(0)}{Y(0)^2} \xi^2 + 2 \tilde{\omega} \right] A \approx 0 \quad . \tag{11}$$

On the ξ distance scale A must vanish at "infinity," and we have a well known eigenvalue problem arising in the quantum theory of the harmonic oscillator. The lowest eigenfunction is the Gaussian $A = \exp \left\{ - \frac{\pi}{2} Y(0)^{-3/2} \left[- Y''(0) \right]^{1/2} \xi^2 \right\}$ with real eigenvalue $\tilde{\omega} = \frac{1}{2} \left[- Y''(0) / Y(0) \right]^{1/2}$.

Incidentally, if we should be interested in the behavior of u for $| x |$ not very small, where u decreases rapidly, a different procedure must be used. The right side of (8) cannot be neglected there, since

$\omega \approx \pi/Y(0)$ does not even approximate the local eigenvalue $\pi/Y(x)$ for which the left side can vanish with $u \neq 0$. The device mentioned earlier of representing the unknown as an exponential works here; with $u = \exp v$, (8) becomes

$$\frac{\partial^2 v}{\partial \eta^2} + \left(\frac{\partial v}{\partial \eta}\right)^2 + \omega^2 = -\epsilon^2 \left[\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right] \quad . \tag{12}$$

We may assume that v is expandable as a series in ϵ, $v = \epsilon^{-1} [v^{(o)} + \epsilon v^{(1)} + \ldots]$, where the leading term has been taken large of order ϵ^{-1} to permit the right side of (12) to contribute. We must have $\frac{\partial v^{(o)}}{\partial \eta} = 0$ or the left side will dominate again, so $v^{(o)}$ is a function of x only, and to dominant terms (12) becomes

$$\frac{\partial^2 v^{(1)}}{\partial \eta^2} + \left(\frac{\partial v^{(1)}}{\partial \eta}\right)^2 + \omega_o^2 = -\left(\frac{\partial v^{(o)}}{\partial x}\right)^2 \quad .$$

Viewed as an equation for $v^{(1)}$ this can be linearized and "homogenized" by reversing the exponentiation procedure, namely by introducing $w = \exp v^{(1)}$, whence

$$\frac{\partial^2 w}{\partial \eta^2} + \left[\omega_o^2 + \left(\frac{\partial v^{(o)}}{\partial x}\right)^2\right] w = 0 \quad .$$

Together with the boundary conditions on w (that it vanish at $\eta = 0$, $Y(x)$) this is an eigenvalue problem which determines the variation of $v^{(o)}$,