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ASYMPTOTOLOGY*

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When I first saw the program for this conference I was mildly curious about why my talk was scheduled at the end of the first session, following the opening lecture by Professor Grad. Although accepted conference manners (conventional convention conventions, I almost said) forbade inquiring of our genial organizers, I now know the reason--Harold's stimulating and excellent lecture has roused a furore of excitement and even controversy, as they must have foreseen, and it is my function to calm you down, bore you perhaps, and send you off properly soothed and relaxed to enjoy tonight's banquet.

The subject of this conference is unusual, and if I am not at all confident that my chosen topic is entirely appropriate, I am emboldened to proceed because of a conviction that it would be out of place anywhere else. But I do feel some trepidation at having Professor Friedrichs in the audience, since I am so heavily indebted to his most enlightening 1955 Gibbs Lecture article,[1] already referred to by Grad.

Asymptotics is the science which deals with such questions as the asymptotic evaluation of integrals, of solutions of differential equations, etc., in various limiting cases. Elements of this science

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* This text is based on a lecture I presented at the Conference on Mathematical Models in Physics at Notre Dame, April 15-17, 1962.

may be learned from the works of van der Corput,[2] Erdélyi,[3] and de Bruijn,[4] and advanced aspects from the numerous references in Friedrichs' cited article. By asymptotology I mean something much broader than asymptotics, but including it; pending further elaboration, I would briefly define asymptotology as the art of dealing with applied mathematical systems in limiting cases.

The first point to note here is that asymptotology is an art, at best a quasi-science, but not a science. Indeed, this explains much of my difficulty both in expounding my material and in finding an appropriate occasion to do so, and it may serve handily to excuse my effort for lacking the high degree of polish which Dean Rossini in his opening remarks assured us we may expect of the presentations (and indeed there does seem to be much Polish about this conference). It explains, too, why I am unable to support the corpus of my dissertation with the hard bones of theorems but must be content with a cartilage of principles, into seven of which I have distilled whatever of asymptotology I have been able to formulate appropriately and sufficiently succinctly.

The aspect of the definition of asymptotology just given which is most in need of explanation is the concept of applied mathematical system. An applied mathematical system is merely the mathematical description of a physical (or occasionally biological or other) system in which the variables expressing the state of the system are complete. The importance of formulating problems in terms of complete state variables constitutes a preliminary principle, not particularly of asymptotology but of applied mathematics in general, the Principle of Classification (or, perhaps better, of Determinism). It is illustrated by the overpowering tendency, in treating classical mechanical problems, to enlarge the configuration space to a phase space, since the phase (configuration together with its rate of change) but not the configuration alone constitutes a complete description of a classical mechanical system. Consider also the tendency, in treating probabilistic mechanical problems, to switch over from this original description, which is incomplete because, for instance, the mechanical "state" at one time does not determine the "state" at another time, to a new description in terms of a probability distribution function of the old "states," which function evolves "deterministically" in time and is therefore preferable as a state description. This Principle is obviously closely related to the notion of a well posed problem emphasized by Hadamard. Its particular relevance to asymptotology comes about because only after one has singled out ("determined") an individual solution (or completely "classified" the family of solutions) can one reasonably inquire into its asymptotic behavior.

Asymptotology is important because the examination of limiting cases seems to be the only satisfactory effective method of proceeding with the analysis of complicated problems (systems) when exact mathematical methods are of no (further) avail (and is often preferable even when they are). It is of value both for obtaining qualitative information (insight) about the behavior of a system and its solutions and for obtaining detailed quantitative (numerical) results. Thus it is hardly surprising that examples, from trivial ones to the most profound, are found everywhere throughout the fields to which analysis (in the technical sense as a branch of mathematics) is applied.

An excellent example of asymptotology is the familiar Hilbert[5] or Chapman-Enskog[6] ("HCE" from now on) theory of a gas described by the Boltzmann equation

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial f}{\partial \underset{\sim}{x}} + \underset{\sim}{a} \cdot \frac{\partial f}{\partial \underset{\sim}{v}} = \lambda \int d^3 v \, d\Omega \, |\underset{\sim}{v} - \underset{\sim}{v}'| \sigma \, [\, \bar{f} \bar{f}' - f f' \,] \qquad (1)$$

in the limit of high density $(f \rightarrow \infty)$ or equivalently of frequent collisions $(\lambda \rightarrow \infty)$, which Grad has already discussed this afternoon. Another example is the Chew-Goldberger-Low[7] theory of the so-called Vlasov[8] system of equations governing an ideal collisionless plasma and its electromagnetic field in what is often called the strong magnetic field (or small gyration radius) limit but is formally best treated[9] as the limit of large particle charges. In the general theory of relativity there is the fundamental Einstein-Infeld-Hoffman[10] derivation of the equation of motion of a "test particle" (one not influencing the space-time metric, i.e. one of negligible mass) by treating it (its world-line, rather) as an appropriate singularity in the metric and letting the strength of the singularity approach zero. Hydrodynamics is rich in asymptotology (theory of shocks as arising in the limit of small viscosity and heat conductivity, theories of strong shocks and of weak shocks, shallow water theory, and so on and on), and so is elasticity. Kirchoff's laws for electrical circuits can be properly derived from Maxwell's equations only by going to the limit of infinitely thin conductors (wires). Simpler examples also abound and are encountered daily by the practicing applied mathematician and theoretical physicist. Naturally it is not practical to discuss deep examples in detail here, so I shall have to confine myself to brief remarks about them, relying for illustration mainly on simple and often trivial instances.

It should now be apparent, I hope, that whatever features such important, wide-spread, and diverse examples may have in common, and whatever lessons for future application may be gleaned from studying them, are well worth formulating and eventually standardizing. Even the many (most? far from all, as I know from my acquaintance) applied mathematicians (etc.) who have become familiar by experience with asymptotological principles, at least in the sense of knowing how to apply them in practice, --even they must inevitably benefit from the introduction of a standard terminology and of the clarity of expression it permits. Implicit knowledge, no matter how widely distributed, deserves explicit formulation, but I am aware of no efforts in this direction which attempt to go anything like so far as I am doing here, though there are some related suggestions in Friedrichs' article.

The final possible obscurity in our previous tentative definition of asymptotology is what it means to deal with a system. To clarify this, we might alternatively define asymptotology as the art of describing the behavior of a specified solution (or family of solutions) of a system in a limiting case. And the answer quite generally has the form of a new system (well posed problem) for the solution to satisfy, although this is sometimes obscured because the new system is so easily solved that one is led directly to the solution without noticing the intermediate step.

To illustrate first by a trivial example, suppose it is desired to follow the (algebraically) largest root x of the simple polynomial equation

$$3\epsilon^2 x^3 + x^2 - \epsilon x - 4 = 0 \qquad (2)$$

in the limit $\epsilon \rightarrow 0$. There is one root of order ϵ^{-2} obtained by treating the first two terms as dominant, $x \approx -\frac{1}{3}\epsilon^{-2}$, for which indeed the other two terms are relatively negligible (even though one of them is absolutely large, of order ϵ^{-1}), but which is negative. The other two roots are finite, obtained by neglecting the terms with ϵ factors, $x \approx \pm 2$, the one sought having the plus sign. If we desire it to higher order, incidentally, we may put (2) for this root in the "recursion" form

$$x = 2\left(1 - \frac{3}{4}\epsilon^2 x^3 + \frac{1}{4}\epsilon x\right)^{1/2} , \tag{3}$$

expand out the right-hand side in powers of ϵ, and generate better and better approximations for x by continually substituting the previously best approximation into the right side. But this is irrelevant to the present point, which is that (the problem of the algebraically largest root of) the original cubic equation (2) has been replaced by (the problem of the algebraically largest root of) the quadratic equation $x^2 - 4 \approx 0$, or more exactly $x^2 - (4 - 3\epsilon^2 x^3 + \epsilon x) = 0$, the quantity in parentheses being treated as known.

In the HCE treatment of system (1) in the limit $\lambda \to \infty$, the original integro-differential equation in the seven independent variables t, $\underset{\sim}{x}$, $\underset{\sim}{v}$ gets replaced by the set of coupled partial differential (hydrodynamic) equations

$$\frac{\partial \rho}{\partial t} \approx - \frac{\partial}{\partial \underset{\sim}{x}} \cdot (\rho \underset{\sim}{u}) ,$$

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \frac{\partial}{\partial \underset{\sim}{x}} \underset{\sim}{u} \approx - \frac{1}{\rho} \frac{\partial p}{\partial x} , \tag{4}$$

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{u} \cdot \frac{\partial}{\partial \underset{\sim}{x}}\right) (\rho^{-5/3} p) \approx 0$$

in the four independent variables t, $\underset{\sim}{x}$; here ρ, $\underset{\sim}{u}$, p are of course the usual velocity space moments of f.

These examples clearly illustrate the first asymptotological principle, which is in fact largely the raison d'être of asymptotology. This Principle of Simplification states that an asymptotological (limiting) analysis tends to simplify the system considered. This can occur in at least three general ways.

The basic way systems simplify is merely by the neglect of terms (or, in higher order analyses, at least treatment of small terms as if known, as in the case of the cubic equation earlier). Thus the polynomial equations $x^5 + \epsilon x + 1 = 0$ and $x^6 + a x^4 + \epsilon x^3 + 1 = 0$, without getting lower in degree as the cubic did, nevertheless become simple enough in the limit $\epsilon \to 0$ to be explicitly solvable algebraically. Differential equations in irregular domains approximating regular ones may in the limit become solvable by separation of variables. In other cases the coefficients may become so simple in the limit as to permit solution by Fourier or other transform. These are typical instances of perturbation theory; there are of course also many instances where the simplification which occurs does not appreciably facilitate the further analysis of the system.

A derivative way in which systems simplify, sometimes striking in effect, is the decomposition of the system into two or more independent systems among which the solutions are divided, so that the particular solution of interest satisfies a system with fewer solutions and hence usually in some sense of lower order. Thus the cubic polynomial equation considered earlier split up into a quad-