"Annihilators" in Asymptotic Methods

(Ref: M. D. Kruskal, Asymptotology)

- Suppose Vector space $\{x\}$

  Operator $\mathbb{B}$

- Suppose $\mathbb{B}x = y$, y given, find x.

Moral

- x does not always exist unless y satisfies certain conditions

- Vector Space proof

  Suppose $\mathbb{B}$ is Hermitean

  [ie, $(\alpha, \mathbb{B}\beta) = (\mathbb{B}\alpha, \beta)$]

  Suppose $\mathbb{B}h_n = 0$, ie h_n are solns to homogeneous eqn

Then, $Bx = y \Rightarrow (h_n, Bx) = (h_n, y)$

$\Rightarrow (Bh_n, x) = (h_n, y) \Rightarrow 0 = (h_n, y)$

$\therefore$ x exists only if $(h_n, y) = 0 \ \forall n$

- <u>More general proof for <u>any</u> B</u>

Suppose we can find an "annihilator" A, such that $A(\text{LHS}) = 0$,

ie, $A(Bx) = 0$.

Then, $Bx = y \Rightarrow ABx = Ay$

$\Rightarrow 0 = Ay$.

$\therefore$ x exists only if $Ay = 0 \ \forall A$ annihilators

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<u>Bottomline</u> : $Bx = y$, y given,

$\Rightarrow$ x exists only if $Ay = 0$

where A is any annihilator for LHS.

Trivial example

Suppose $\vec{c} \times \vec{x} = \vec{y}$, $\vec{y}$ given,
find $\vec{x}$.

Annihilator clearly $\{\vec{c}\,\cdot\}$
annihilates LHS since

$$\vec{c} \cdot (\vec{c} \times \vec{x}) = 0$$

$\therefore$ $\vec{x}$ exists provided $\vec{c} \cdot \vec{y} = 0$.

More complicated example — see Appendix in Notes